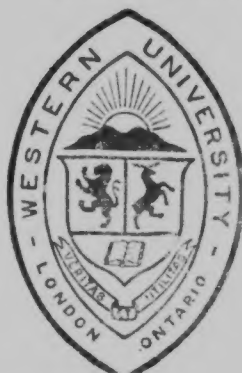




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EUCLID'S BOOK  
ON DIVISIONS OF FIGURES

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# EUCLID'S BOOK ON DIVISIONS OF FIGURES

(περὶ διαρέσεων βιβλίον)

WITH A RESTORATION BASED ON  
WOEPCKE'S TEXT

AND ON THE

*PRACTICA GEOMETRIÆ*  
OF LEONARDO PISANO

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1317

TO  
MY OLD TEACHER AND FRIEND  
ALFRED DEANE SMITH  
PROFESSOR OF GREEK AND LATIN  
AT MOUNT ALLISON UNIVERSITY  
FOR FORTY-FOUR YEARS  
SCHOLAR OF GREAT ATTAINMENTS  
THE WONDER OF ALL WHO KNOW HIM  
THESE PAGES ARE AFFECTIONATELY DEDICATED

## INTRODUCTORY

**E**UCLID, famed founder of the Alexandrian School of Mathematics, was the author of not less than nine works. Approximately complete texts, all carefully edited, of four of these, (1) the *Elements*, (2) the *Data*, (3) the *Optics*, (4) the *Phenomena*, are now our possession. In the case of (5) the *Pseudaria*, (6) the *Surface-Loci*, (7) the *Conics*, our fragmentary knowledge, derived wholly from Greek sources, makes conjecture as to their content of the vaguest nature. On (8) the *Porisms*, Pappus gives extended comment. As to (9), the book *On Divisions (of figures)*, Proclus alone among Greeks makes explanatory reference. But in an Arabian MS., translated by Woepcke into French over sixty years ago, we have not only the enunciations of all of the propositions but also the proofs of four of them.

Whilst elaborate restorations of the *Porisms* by Simson and Chasles have been published, no previous attempt has been made (the pamphlet of Osterdinger is not forgotten) to restore the proofs of the book *On Divisions (of figures)*. And, except for a short sketch in Heath's monumental edition of Euclid's *Elements*, nothing but passing mention of Euclid's book *On Divisions* has appeared in English.

In this little volume I have attempted :

(1) to give, with necessary commentary, a restoration of Euclid's work based on the Woepcke text and on a thirteenth century geometry of Leonardo Pisano.

(2) to take due account of the various questions which arise in connection with (a) certain MSS. of "Muhammed Bagdedinus," (b) the Dee-Commandinus book on divisions of figures.

(3) to indicate the writers prior to 1500 who have dealt with propositions of Euclid's work.

(4) to make a selection from the very extensive bibliography of the subject during the past 400 years.

In the historical survey the MSS. of "Muhammed Bagdedinus" play an important rôle, and many recent historians, for example Heiberg, Cantor, Hankel, Loria, Suter, and Steinschneider, have contributed to the discussion. As it is necessary for me to correct errors, major and minor, of all of these writers, considerable detail has to be given in the first part of the volume; the brief second part treats of writers on divisions before 1500; the third part contains the restoration proper, with its thirty-six propositions. The Appendix deals with literature since 1500.

A score of the propositions are more or less familiar as isolated problems of modern English texts, and are also to be found in many recent English, German and French books and periodicals. But any approximately accurate restoration of the work as a whole, in Euclidean manner, can hardly fail of appeal to anyone interested in elementary geometry or in Greek mathematics of twenty-two centuries ago.

In the spelling of Arabian names, I have followed Suter.

It is a pleasure to have to acknowledge indebtedness to the two foremost living authorities on Greek Mathematics. I refer to Professor J. L. Heiberg of the University of Copenhagen and to Sir Thomas L. Heath of London. Professor Heiberg most kindly sent me the proof pages of the forthcoming concluding volume of Euclid's *Opera Omnia*, which contained the references to Euclid's book *On Divisions of Figures*. To Sir Thomas my debt is great. On nearly every page that follows there is evidence of the influence of his publications; moreover, he has read this little book in proof and set me right at several points, more especially in connection with discussions in Note 113 and Paragraph 50.

R. C. A.

BROWN UNIVERSITY,

June, 1915.

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# I.

## *Proclus, and Euclid's book On Divisions.*

1. Last in a list of Euclid's works "full of admirable diligence and skilful consideration," Proclus mentions, without comment, *περὶ διαιρέσεων βιβλίον*<sup>1</sup>. But a little later<sup>2</sup> in speaking of the conception or definition of *figure* and of the divisibility of a figure into others differing from it in kind, Proclus adds: "For the circle is divisible into parts unlike in definition or notion, and so is each of the rectilineal figures; this is in fact the business of the writer of the *Elements* in his *Divisions*, where he divides given figures, in one case into like figures, and in another into unlike<sup>3</sup>."

## *De Divisionibus by Muhammed Bagdedinus and the Dec MS.*

2. This is all we have from Greek sources, but the discovery of an Arabian translation of the treatise supplies the deficiency. In histories of Euclid's works (for example

<sup>1</sup> *Procli Diadochi in primum Euclidis elementorum librum commentarii* ex rec. G. Friedlein, Leipzig, 1873, p. 69. Reference to this work will be made by "Proclus."

<sup>2</sup> Proclus<sup>1</sup>, p. 144.

In this translation I have followed T. L. HEATH, *The Thirteen Books of Euclid's Elements*, 1. Cambridge, 1908, p. 8. To Heath's account pp. 8-10 of Euclid's book *On Divisions* I shall refer by "Heath."

"Like" and "unlike" in the above quotation mean, not "similar" and "dissimilar" in the technical sense, but "like" or "unlike in definition or notion": thus to divide a triangle into triangles would be to divide it into "like" figures, to divide a triangle into a triangle and a quadrilateral would be to divide it into "unlike" figures. Heath.



stated by Steinschneider when he writes in 1905<sup>14</sup>, "Machomet Bagdadinus (= aus Bagdad) heisst in einem alten MS. Cotton (jetzt im Brit. Mus.) der Verfasser von: de Superficierum divisione (22 Lehrsätze); Jo. Dee aus London entdeckte es und übergab es T. Commandino...." For this suggestion as to the place where Dee found the MS. Steinschneider gives no authority. He does, however, give a reference to Wenrich<sup>15</sup>, who in turn refers to a list of the printed books ("Impressi") of John Dee, in a life of Dee by Thomas Smith<sup>16</sup> (1638-1710). We here find as the third in the list, "Epistola ad eximium Ducis Urbini Mathematicum, Fredericum Commandinum, praefixa libello Machometi Bagdedini de superficierum divisionibus...Pisauri, 1570. Exstat MS. in Bibliotheca Cottoniana sub Tiberio B ix."

Then come the following somewhat mysterious sentences which I give in translation<sup>17</sup>: "After the preface Lord Ussher [1581-1656], Archbishop of Armagh, has these lines: It is to be noted that the author uses Euclid's Elements translated into the Arabic tongue, which Campanus afterwards turned into Latin. Euclid therefore seems to have been the author of the Propositions [of *De Divisionibus*] though not of the demonstrations, which contain references to an Arabic edition of the Elements, and which are due to Machometus of Bagded or Babylon." This quotation from Smith is reproduced, with various changes in punctuation and typography, by Kästner<sup>18</sup>. Consideration of the latter part of it I shall postpone to a later article (5).

<sup>14</sup> M. STEINSCHNEIDER, "Die Europäischen Übersetzungen aus dem Arabischen bis Mitte des 17. Jahrhunderts." *Sitzungsberichte der Akademie der Wissenschaften in Wien Phil.-histor. Klasse* CL, Jan. 1905, Wien, 1906. Concerning "171. Muhammed" cf. pp. 41-2. Reference to this paper will be made by "Steinschneider."

<sup>15</sup> J. G. WENRICH, *De auctorum Graecorum versionibus*. Lipsiae, MDCCCLX, p. 184.

<sup>16</sup> T. SMITH, *Vitae quorundam eruditissimorum et illustrium virorum...* Londini...MDCCVII, p. 56. It was only the first 55 pages of this "Vita Johannis Dee, Mathematici Angli," which were translated into English by W. A. Ayton, London, 1906.

<sup>17</sup> "Post praefationem hanc habet D. Usserius Archiepiscopus Armachanus: *Notandum est autem, Auctorem hunc Euclide usum in Arabicam linguam conversum, quem postea Campanus Latinum fecit. Auctor igitur propositionum videtur fuisse Euclides; demonstrationum, in quibus Euclides in Arabico codice citatur, Machometus Bagded sive Babylonius.*

It has been stated that Campanus (13. cent.) did not translate Euclid's Elements into Latin, but that the work published as his Venice, 1482—the first printed edition of the *Elements*—was the translation made about 1120 by the English monk Athelhard of Bath. Cf. HEATH, *Thirteen Books of Euclid's Elements*, I, 78, 93-96.

<sup>18</sup> A. G. KÄSTNER, *Geschichte der Mathematik...* Erster Band...Göttingen, 1797, pp. 272-3. See also "Zweyter" Band, 1797, pp. 46-47.

3. Following up the suggestion of Steinschneider, Suter pointed out<sup>17</sup>, without reference to Smith<sup>14</sup> or Kästner<sup>16</sup>, that in Smith's catalogue of the Cottonian Library there was an entry<sup>18</sup> under "Tiberius" B. ix, 6": "*Liber Divisionum Mahumeti Bag-dadini*." As this MS. was undoubtedly in Latin and as Cottonian MSS. are now in the British Museum, Suter inferred that Dee simply made a copy of the above mentioned MS. and that this MS. was now in the British Museum. With his wonted carefulness of statement, Heath does not commit himself to these views although he admits their probable accuracy.

4. As a final settlement of the question, I propose to show that Steinschneider and Suter, and hence also many earlier writers, have not considered all facts available. Some of their conclusions are therefore untenable. In particular:

(1) In or before 1563 Dee did *not* make a copy of any Cottonian MS.;

(2) The above mentioned MS. (*Tiberius*, B. ix, 6) was *never*, in its entirety, in the British Museum;

(3) The inference by Suter that this MS. was probably the Latin translation of the tract from the Arabic, made by Gherard of Cremona (1114-1187)—among the lists of whose numerous translations a "*liber divisionum*" occurs—*should be accepted with great reserve*;

(4) The MS. which Dee used can be stated with absolute certainty and this MS. did not, in all probability, afterwards become a Cottonian MS.

(1) Sir Robert Bruce Cotton, the founder of the Cottonian Library, was born in 1571. The Cottonian Library was not, therefore, in existence in 1563 and Dee could not then have copied a Cottonian MS.

(2) The Cottonian Library passed into the care of the nation shortly after 1700. In 1731 about 200 of the MSS.

<sup>17</sup> H. SUTER, "Zu dem Buche 'De Superficierum divisionibus' des Muhammed Bagdedinus," *Bibliotheca Mathematica*, VI, 321-2, 1905.

<sup>18</sup> F. SMITH, *Catalogus Librorum Manuscriptorum Bibliothecae Cottonianae*, Oxonii... MDCXCVI, p. 24.

<sup>19</sup> The original Cottonian library was contained in 14 presses, above each of which was a bust: 12 of these busts were of Roman Emperors. Hence the classification of the MSS. in the catalogue.

were damaged or destroyed by fire. As a result of the parliamentary inquiry Casley reported<sup>1</sup> on the MSS. destroyed or injured. Concerning Tiberius IX, he wrote, "This volume burnt to a crust." He gives the title of each tract and the folios occupied by each in the volume. "Liber Divisionum Mahumeti Bag-dadini" occupied folios 254-258. When the British Museum was opened in 1753, what was left of the Cottonian Library was immediately placed there. Although portions of all of the leaves of our tract are now to be seen in the British Museum, practically none of the writing is decipherable.

(3) Planta's catalogue<sup>2</sup> has the following note concerning Tiberius IX: "A volume on parchment, which once consisted of 272 leaves, written about the XIV. century [not the XII. century, when Gherard of Cremona flourished], containing eight tracts, the principal of which was a 'Register of William Cratfield, abbot of St. Edmund'" [d. 1415]. Tracts 3, 4, 5 were on music.

(4) On "A" 1583, 6 Sept." Dee made a catalogue of the MSS. which he owned. This catalogue, which is in the Library of Trinity College, Cambridge<sup>3</sup>, has been published:

<sup>1</sup> D. CASLEY, p. 15 ff. of *A Report from the Committee appointed by the House of Commons, London, 1753*. Cf. also the page opposite that numbered 120 in *A Catalogue of the Manuscripts in the Cottonian Library... with an Appendix containing an account of the damage sustained by the Fire in 1731*. London: 1753.

<sup>2</sup> J. PLANTA, *A Catalogue of the Manuscripts in the Cottonian Library deposited in the British Museum. Printed by command of his Majesty King George III...* 1802.

In the British Museum there are three MS. catalogues of the Cottonian Library.

<sup>3</sup> *Harleian MS.* 6018, a catalogue made in 1621. At the end are memoranda of loaned books. On a sheet of paper bearing date Novem. 23. 1638, Tiberius B IX is listed folio 187 with its art. 4: "Liber divisione Machumeti Bagdedini. The paper is torn so that the name of the person to whom the work was loaned is missing. The volume is not mentioned in the main catalogue.

<sup>4</sup> *MS.* Vol. 36789, made after Sir Robert Cotton's death in 1631 and before 1638. Cf. *Catalogue of Additions to the MSS. in British Museum, 1600-1605...* London 1907, pp. 226-227, contains, apparently, no reference to "Muhammed".

<sup>5</sup> *MS.* Vol. 36082 A, of uncertain date but earlier than 1654. *Catalogue of Additions...* pp. 188-189. On folio 78 verso we find Tiberius B IX. Art. 4: "Liber divisione Machumeti Bagdedini".

A "Muhammed" MS. was therefore in the Cottonian Library in 1631.

The anonymously printed 1840? "Index to articles printed from the Cotton MSS., & where they may be found" which may be seen in the British Museum, only gives references to the MSS. in "Index".

A transcription of the Trinity College copy, by Ashmole, is in *MS.* Ashm. 1112. Another autograph copy is in the British Museum: *Harleian MS.* 187.

<sup>6</sup> *Cottonian MS.* Vol. 36789, *MS.* Vol. 36082 A, *MS.* Vol. 36082 B.

under the editorship of J. O. Halliwell. The 95th item described is a folio parchment volume containing 24 tracts on mathematics and astronomy. The 17th tract is entitled "Machumeti Bagdedini liber divisionum." As the contents of this volume are entirely different from those of Tiberius ix described above, in (3), it seems probable that there were two copies of "Muhammed's" tract, while the MS. which Dee used for the 1570 publication was undoubtedly his own, as we shall presently see. If the two copies be granted, there is no evidence against the Dee copy having been that made by Gherard of Cremona.

5. There is the not remote possibility that the Dee MS. was destroyed soon after it was catalogued. For in the same month that the above catalogue was prepared, Dee left his home at Mortlake, Surrey, for a lengthy trip in Europe. Immediately after his departure "the mob, who execrated him as a magician, broke into his house and destroyed a great part of his furniture and books<sup>1</sup>... many of which "were the written bookes<sup>2</sup>." Now the Dee catalogue of his MSS. (MS. O. iv. 20), in Trinity College Library, has numerous annotations<sup>3</sup> in Dee's handwriting. They indicate just what works were (1) destroyed or stolen ("Fr.")<sup>4</sup> and (2) left ("T.")<sup>5</sup> after the raid. Opposite the titles of the tracts in the volume including the tract "liber divisionum," "Fr." is written, and opposite the title "Machumeti Bagdedini liber divisionum" is the following note: "Curavi imprimi Urbini in Italia per Federicum Commandinum exemplari descripto ex vetusto isto monumento (?) per me ipsum." Hence, as stated above, it is now definitely known (1) that the MS. which Dee used was his own, and (2) that some 20 years after he made a copy, the MS. was stolen and probably destroyed<sup>6</sup>.

On the other hand we have the apparently contradictory

<sup>1</sup> *Dictionary of National Biography*, Article, "Dee, John."

<sup>2</sup> "The compendious rehearsall of John Dee his dutifull declaration A. 1592 printed in *Chetham Miscellanies*, vol. 4, Manchester, 1851, p. 27.

<sup>3</sup> Although Halliwell professed to publish the Trinity MS., he makes not the slightest reference to these annotations.

<sup>4</sup> "Fr." is no doubt an abbreviation for *Fractum*.

<sup>5</sup> "T.", according to Ainsworth *Latin Dictionary*, was put after the name of a soldier to indicate that he had survived *superstes*. Whence this abbreviation?

<sup>6</sup> The view concerning the theft or destruction of the MS. is borne out by the fact that in a catalogue of Dee's Library (British Museum MS. 35213<sup>7</sup> made early in the seventeenth century (*Catalogue of Additions and Manuscripts*... 1901, p. 211), Machumeti Bagdedini is not mentioned.

evidence in the passage quoted above (Art. 2) from the life of Dee by Smith<sup>1</sup> who was also the compiler of the Catalogue of the Cottonian Library. Smith was librarian when he wrote both of these works, so that any definite statement which he makes concerning the library long in his charge is not likely to be successfully challenged. Smith does not however say that Dee's "Muhammed" MS. was in the Cottonian Library, and if he knew that such was the case we should certainly expect some note to that effect in the catalogue<sup>2</sup>; for in three other places in his catalogue (Vespasian B x, A ii, Galba E viii), Dee's original ownership of MSS. which finally came to the Cottonian Library is carefully remarked. Smith does declare, however, that the Cottonian MS. bore, "after the preface," certain notes (which I have quoted above) by Archbishop Ussher (1581-1656). Now it is not a little curious that these notes by Ussher, who was not born till after the Dee book was printed, should be practically identical with notes in the printed work, just after Dee's letter to Commandinus (Art. 3). For the sake of comparison I quote the notes in question: "To the Reader.—I am here to advertise thee (kinde Reader) that this author which we present to thee, made use of Euclid tranflated into the Arabick Tongue, whom afterwards Campanus made to speake Latine. This I thought fit to tell thee, that so in searching or examining the Propofitions which are cited by him, thou mightest not sometime or other trouble thy selfe in vain, Farewell."

The Dee MS. as published did not have any preface. We can therefore only assume that Ussher wrote in a MS. which *did* have a preface the few lines which he may have seen in Dee's printed book.

6. Other suggestions which have been made concerning "Muhammed's" tract should be considered. Steinschneider asks, "Ob identisch de Curvis superficiebus, von einem Muhammed, MS. Brit. Mus. Harl. 623<sup>6</sup> (I, 191)<sup>1</sup>?" I have examined this MS. and found that it has nothing to do with the subject matter of the Dee tract.

But again, Favaro states<sup>2</sup>: "Probabilmente il manoscritto

<sup>6</sup> This quotation from the Leake-Serle Euclid<sup>11</sup> is an exact translation of the original.

<sup>1</sup> This should be 625<sup>6</sup> I, 391.

<sup>2</sup> Favaro, p. 140. Cf. Heiberg, p. 14. This suggestion doubtless originated with Osterdinger<sup>38</sup>, p. [1].





only on the propositions (or easy deductions from them) of the *Elements*, while Woepeke 18 has the true Greek ring: to apply to a straight line a rectangle equal to the rectangle contained by  $AB$ ,  $AC$  and *deficient by a square*.

8. To no proposition in the Dee MS. is there word for word correspondence with the propositions of Woepeke but in content there are several cases of likeness. Thus, Heiberg continues:

Dee 3 = Woepeke 30 (a special case is Woepeke 11).

Dee 7 = Woepeke 34 (a special case is Woepeke 14).

Dee 9 = Woepeke 36 (a special case is Woepeke 16);

Dee 12 = Woepeke 32 (a special case is Woepeke 10).

Woepeke 3 is only a special case of Dee 2; Woepeke 6, 7, 8, 9 are easily solved by Dee 8. And it can hardly be chance that the proofs of exactly these propositions in Dee should be without fault. That the treatise published by Woepeke is no fragment but the complete work which was before the translator is expressly stated<sup>7</sup>, "fin du traité." It is moreover a well ordered and compact whole. Hence we may safely conclude that Woepeke's is not only Euclid's own work but the whole of it, except for proofs of some propositions.

9. For the reason just stated the so-called *Wiederherstellung* of Euclid's work by Offerdinger<sup>8</sup>, based mainly on Dee, is decidedly misnamed. A more accurate description of this pamphlet would be, "A translation of the Dee tract with indications in notes of a certain correspondence with 15 of Woepeke's propositions, the whole concluding with a translation of the enunciations of 16 of the remaining 21 propositions of Woepeke not previously mentioned." Woepeke 30, 31, 34, 35, 36 are not even noticed by Offerdinger. Hence the claim I made above ("Introductory") that the first real restoration of Euclid's work is now presented. Having introduced Woepeke's text as one part of the basis of this restoration, the other part demands the consideration of the

#### *Practica Geometriae of Leonardo Pisano (Fibonacci).*

10. It was in the year 1220 that Leonardo Pisano, who occupies such an important place in the history of mathematics

<sup>7</sup> Woepeke, p. 214.

<sup>8</sup> F. F. Offerdinger, *Die Wiederherstellung des Euclid'schen Textes*, *Publ. of the Math. Inst. of the Univ. of Bonn*, 1933.

of the thirteenth century", wrote his *Practica Geometriae*, and the MS. is now in the Vatican Library. Although it was known and used by other writers, nearly six and one half centuries elapsed before it was finally published by Prince Boncompagni<sup>10</sup>. Favaro was the first<sup>11</sup> to call attention to the importance of Section IIII<sup>12</sup> of the *Practica Geometriae* in connection with the history of Euclid's work. This section is wholly devoted to the enunciation and proof and numerical exemplification of propositions concerning the divisions of figures. Favaro reproduces the enunciations of the propositions and numbers them 1 to 57<sup>13</sup>. He points out that in both enunciation and proof Leonardo 3, 10, 51, 57 are identical with Woepeke 19, 20, 29, 28 respectively. But considerably more remains to be remarked.

II. No less than twenty-two of Woepeke's propositions are practically identical in statement with propositions in Leonardo; the solutions of eight more of Woepeke are either given or clearly indicated by Leonardo's methods, and all six of the remaining Woepeke propositions (which are auxiliary) are assumed as known in the proofs which Leonardo gives of propositions in Woepeke. Indeed, these two works have a remarkable similarity. Not only are practically all of the Woepeke propositions in Leonardo, but the proofs called for by the order of the propositions and by the auxiliary propositions in Woepeke are, with a possible single exception<sup>14</sup>, invariably the kind of proofs which Euclid might have given — no other propositions but those which had gone before or which were to be found in the *Elements* being required in the successive constructions.

Leonardo had a wide range of knowledge concerning Arabian mathematics and the mathematics of antiquity. His *Practica Geometriae* contains many references to Euclid's *Elements* and many uncredited extracts from this work<sup>15</sup>.

<sup>10</sup> M. CANTOR, *Lehrbuch der Geschichte der Mathematik*, II, 1900, pp. 3-53.

<sup>11</sup> *Practica Geometriae*, pp. 35-40.

<sup>12</sup> *Scritti di Leonardo Pisano matematico del secolo decimoterzo pubblicati da Lodovico Boncompagni*. Volume II (Leonardi Pisani *Practica Geometriae* ed opuscoli). Roma...1862. *Practica Geometriae*, pp. 1-224.

<sup>13</sup> *Scritti di Leonardo Pisano*...II, pp. 110-148.

<sup>14</sup> These numbers I shall use in what follows. Favaro omits some auxiliary propositions and makes slips in connection with 28 and 40. Either 28 should have been more general in statement or another number should have been introduced. Similarly for 40. Compare Articles 33-34, 35.

<sup>15</sup> For example, on pages 15-16, 38, 95, 100-1, 154.

Similar treatment is accorded works of other writers. But in the great elegance, finish and rigour of the whole, originality of treatment is not infrequently evident. If Gherard of Cremona made a translation of Euclid's book *On Divisions*, it is not at all impossible that this may have been used by Leonardo. At any rate the conclusion seems inevitable that he must have had access to some such MS. of Greek or Arabian origin.

Further evidence that Leonardo's work was of Greek-Arabic extraction can be found in the fact that, in connection with the 113 figures, of the section *On Divisions*, of Leonardo's work, the lettering in only 58 contains the letters *c* or *f*; that is, the Greek-Arabic succession *a, b, g, d, e, z...* is used almost as frequently as the Latin *a, b, c, d, e, f, g...*; elimination of Latin letters added to a Greek succession in a figure, for the purpose of numerical examples (in which the work abounds), makes the balance equal.

12. My method of restoration of Euclid's work has been as follows. Everything in Woepeke's text (together with his notes) has been translated literally, reproduced without change and enclosed by quotation marks. To all of Euclid's enunciations (unaccompanied by constructions) which corresponded to enunciations by Leonardo, I have reproduced Leonardo's constructions and proofs, with the same lettering of the figures<sup>4</sup>, but occasional abbreviation in the form of statement; that is, the extended form of Euclid in Woepeke's text, which is also employed by Leonardo, has been sometimes abridged by modern notation or briefer statement. Occasionally some very obvious steps taken by Leonardo have been left out but all such places are clearly indicated by explanation in square brackets, [ ]. Unless stated to the contrary, and indicated by different type, no step is given in a construction or proof which is not contained in Leonardo. When there is no correspondence between Woepeke and Leonardo I have exercised care to reproduce Leonardo's methods in other propositions, as closely as possible. If, in a given proposition, the method is extremely obvious on account of what has gone before, I have sometimes given little more than an indication of the propositions containing the essence of the required

<sup>4</sup> This is subject to the following variation on the possibility of a second construction being given. As usual.

construction and proof. In the case of the six auxiliary propositions, the proofs supplied seemed to be readily suggested by propositions in Euclid's *Elements*.

13. Immediately after the enunciations of Euclid's problems follow the statements of the correspondence with Leonardo: if exact, a bracket encloses the number of the Leonardo proposition, according to Favaro's numbering, and the page and lines of Boncompagni's edition where Leonardo enunciates the same proposition.

The following is a comparative table of the Euclid and, in brackets, of the corresponding Leonardo problems: 1 (5); 2 (14); 3 (2, 1); 4 (23); 5 (33); 6 (16); 7 (20)<sup>14</sup>; 8 (27)<sup>15</sup>; 9 (30, 31)<sup>16</sup>; 10 (18); 11 (0); 12 (28)<sup>17</sup>; 13 (32)<sup>18</sup>; 14 (36); 15 (40); 16 (37); 17 (39); 18 (0); 19 (3); 20 (10); 21 (0); 22 (0); 23 (0); 24 (0); 25 (0); 26 (4); 27 (11); 28 (57); 29 (51)<sup>19</sup>; 30 (0); 31 (0); 32 (29); 33 (35); 34 (40)<sup>20</sup>; 35 (0); 36 (0).

#### *Summary.*

It will be instructive, as a means of comparison, to set forth in synoptic fashion: (1) the Muhammed-Commandinus treatise; (2) the Euclid treatise; (3) Leonardo's work. In (1) and (2) I follow Woepcke closely<sup>21</sup>.

#### 14. *Synopsis of Muhammed's Treatise—*

I. In all the problems it is required to divide the proposed figure into two parts having a given ratio.

II. The figures divided are: the triangle (props. 1-6); the parallelogram (11); the trapezium<sup>22</sup> (8, 12, 13); the quadrilateral (7, 9, 14-16); the pentagon (17, 18, 22); a pentagon with two parallel sides (19), a pentagon of which a side is parallel to a diagonal (20).

<sup>14</sup> Leonardo considers the case of "one third" instead of Euclid's "a certain fraction," but in the case of 20 he concludes that in the same way the figure may be divided "into four or many equal parts." Cf. Article 28.

<sup>15</sup> Woepcke 8 may be considered as a part of Leonardo 27 or better as an unnumbered proposition following Leonardo 25.

<sup>16</sup> Leonardo's propositions 30-32 consider somewhat more general problems than Euclid's 9 and 13. Cf. Articles 30 and 34.

<sup>21</sup> Woepcke, pp. 245-24.

## III. The transversal required to be drawn :

A. passes through a given point and is situated :

1. at a vertex of the proposed figure (1, 7, 17);
2. on any side (2, 9, 18);
3. on one of the two parallel sides (8).

B. is parallel :

1. to a side (not parallel) (3, 13, 14, 22);
2. to the parallel sides (11, 12, 19);
3. to a diagonal (15, 20);
4. to a perpendicular drawn from a vertex of the figure to the opposite side (4);
5. to a transversal which passes through a vertex of the figure (5);
6. to any transversal (6, 16).

IV. Prop. 10: Being given the segment  $AB$  and two lines which pass through the extremities of this segment and form with the line  $AB$  any angles, draw a line parallel to  $AB$  from one or the other side of  $AB$  and such as to produce a trapezium of given size.

Prop. 21. Auxiliary theorem regarding the pentagon.

**15.** *Commandinus's Treatise*—Appended to the first published edition of Muhammed's work was a short treatise<sup>49</sup> by Commandinus who said<sup>50</sup> of Muhammed: "for what things the author of the book hath at large comprehended in many problems, I have compendiously comprised and dispatched in two only." This statement repeated by Offerdinger<sup>51</sup> and Favaro<sup>52</sup> is somewhat misleading.

The "two problems" of Commandinus are as follows:

"Problem I. To divide a right lined figure according to a proportion given, from a point given in any part of the ambitus or circuit thereof, whether the said point be taken in any angle or side of the figure."

"Problem II. To divide a right lined figure  $GABC$ ,

<sup>49</sup> Commandinus<sup>11</sup>, pp. 54-5.

<sup>50</sup> Commandinus<sup>11</sup>, p. 57; Locke Scilicet Euclid, p. 605.

<sup>51</sup> Offerdinger<sup>12</sup>, p. 11, note.

Favaro<sup>13</sup>, p. 13.

according to a proportion given,  $E$  to  $F$ , by a right line parallel to another given line  $D$ ."

But the first problem is divided into 18 cases: 4 for the triangle, 6 for the quadrilateral, 4 for the pentagon, 2 for the hexagon and 2 for the heptagon; and the second problem, as Commandinus treats it, has 20 cases: 3 for the triangle, 7 for the quadrilateral, 4 for the pentagon, 4 for the hexagon, 2 for the heptagon.

# 16. *Synopsis of Euclid's Treatise—*

## I. The proposed figure is divided:

1. into two equal parts (1, 3, 4, 6, 8, 10, 12, 14, 16, 19, 26, 28);
2. into several equal parts (2, 5, 7, 9, 11, 13, 15, 17, 29);
3. into two parts, in a given ratio (20, 27, 30, 32, 34, 36);
4. into several parts, in a given ratio (31, 33, 35, 36).

The construction 1 or 3 is always followed by the construction of 2 or 4, except in the propositions 3, 28, 29.

## II. The figures divided are:

- the triangle (1, 2, 3, 19, 20, 26, 27, 30, 31);
- the parallelogram (6, 7, 10, 11);
- the trapezium (4, 5, 8, 9, 12, 13, 32, 33);
- the quadrilateral (14, 15, 16, 17, 34, 35, 36);
- a figure bounded by an arc of a circle and two lines (28);
- the circle (29).

## III. It is required to draw a transversal:

### A. passing through a point situated:

1. at a vertex of the figure (14, 15, 34, 35);
2. on any side (3, 6, 7, 16, 17, 36);
3. on one of two parallel sides (8, 9);
4. at the middle of the arc of the circle (28);
5. in the interior of the figure (19, 20);
6. outside the figure (10, 11, 26, 27);
7. in a certain part of the plane of the figure (12, 13).

B. parallel to the base of the proposed figure (1, 2, 4, 5, 30, 33).

C. parallel to one another, the problem is indeterminate (20).

IV. Auxiliary propositions :

18. To apply to a given line a rectangle of given size and deficient by a square.

21, 22, when  $a : d < b : c$ , it follows that  $a : b > c : d$ ;

23, 24, when  $a : b > c : d$ , it follows that

$$(a \mp b) : b > (c \mp d) : d ;$$

25, when  $a : b < c : d$ , it follows that  $(a - b) : b < (c - d) : d$ .

In the synopsis of the last five propositions I have changed the original notation slightly.

17. *Analysis of Leonardo's Work.* I have not thought it necessary to introduce into this analysis the unnumbered propositions referred to above<sup>a</sup>.

I. The proposed figure is divided :

1. into two equal parts (1-5, 15-18, 23-26, 36-38, 42-46, 53-55, 57) ;

2. into several equal parts (6, 7, 9, 13, 14, 19, 21, 33, 47, 50, 56) ;

3. into two parts in a given ratio (8, 10-12, 20, 29, 32, 34, 39, 40, 51, 52) ;

4. into several parts in a given ratio (22, 35, 41).

The construction 1 or 3 is always followed by the construction of 2 or 4 except in the propositions 42-46, 51, 54, 57.

II. The figures divided are :

the triangle (1-14) ;

the parallelogram (15-22) ;

the trapezium (23-35) ;

the quadrilateral (36-41) ;

the pentagon (42-43) ;

the hexagon (44) ;

the circle and semicircle (45-56) ;

a figure bounded by an arc of a circle and two lines (57).

## III.

(i) It is required to draw a transversal:

A. passing through a point situated:

1. at a vertex of the figure (1, 6, 26, 31, 34, 36, 41-44);
2. on a side not produced (2, 7, 8, 16, 20, 37, 39);
3. at a vertex or a point in a side (40);
4. on one of two parallel sides (24, 25, 27, 30);
5. on the middle of the arc of the circle (53, 55, 57);
6. on the circumference or outside of the circle (45);
7. inside of the figure (3, 10, 15, 17, 46);
8. outside of the figure (4, 11, 12, 18);
9. either inside or outside of the figure (38);
10. either inside or outside or on a side of the figure (32);
11. in a certain part of the plane of the figure (28).

B. parallel to the base of the proposed figure (5, 14, 19, 21, 23, 29, 33, 35, 34);

C. parallel to a diameter of the circle (49, 50).

(ii) It is required to draw more than one transversal (*a*) through one point (9, 47, 48, 56); (*b*) through two points (13); (*c*) parallel to one another, the problem is indeterminate (51).

(iii) It is required to draw a circle (52).

## IV. Auxiliary Propositions:

Although not explicitly stated or proved, Leonardo makes use of four out of six of Euclid's auxiliary propositions<sup>113</sup>. On the other hand he proves two other propositions which Favaro does not number: (1) Triangles with one angle of the one equal to one angle of the other, are to one another as the rectangle formed by the sides about the one angle is to that formed by the sides about the equal angle in the other; (2) the medians of a triangle meet in a point and trisect one another.

## II.

18. *Abraham Savasorda, Jordanus Nemorarius, Luca Pacinolo.*—In earlier articles (10, 11) incidental reference was made to Leonardo's general indebtedness to previous writers in preparing his *Practica Geometriae*, and also to the debt which later writers owe to Leonardo. Among the former, perhaps mention should be made of Abraham bar Chijja ha Nasi<sup>53</sup> of Savasorda and his *Liber embadorum* known through the Latin translation of Plato of Tivoli. Abraham was a learned Jew of Barcelona who probably employed Plato of Tivoli to make the translation of his work from the Hebrew. This translation, completed in 1116, was published by Curtze, from fifteenth century MSS., in 1902<sup>54</sup>. Pages 130-159 of this edition contain "capitulum tertium in arearum divisionum explanatione" with Latin and German text, and among the many other propositions given by Savasorda is that of Proclus-Euclid (=Woepeke 28=Leonardo 57). Compared with Leonardo's treatment of divisions Savasorda's seems rather trivial. But however great Leonardo's obligations to other writers, his originality and power sufficed to make a comprehensive and unified treatise.

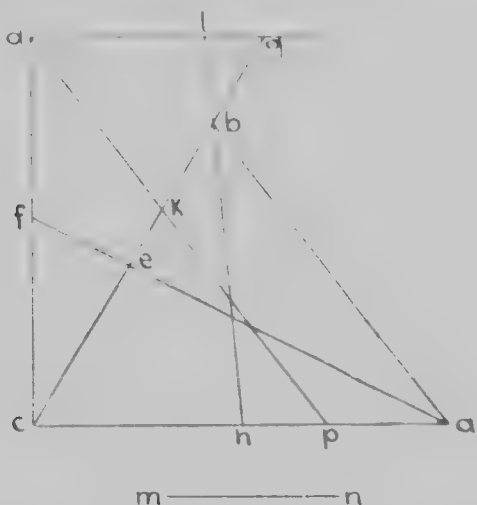
Almost contemporary with Leonardo was Jordanus Nemorarius (d. 1237) who was the author of several works, all probably written before 1222. Among these is *Geometria vel*

<sup>53</sup> That is, Abraham son of Chijja the prince. Cf. STEINSCHNEIDER, *Bibliotheca Mathematica*, 1896, 2, N. 34-38, and CASIOR, *Vorlesungen über Geschichte d. Math.*, I, 797-800, 907.

<sup>54</sup> M. CURTZE, *Die Entwicklung der Geometrie in der Mathematik im Mittelalter und der Renaissance*. Festschrift, Leipzig, 1902, pp. 3-153.

*De Triangulis*\* in four books. The second book is principally devoted to problems on divisions: Propositions 1-7 to the division of lines and Propositions 8, 13, 17, 18, 19 to the division of rectilinear figures. The enunciations of Propositions 8, 13, 17, 19 correspond, respectively, to Euclid 3, 26, 19, 14 and to Leonardo 2, 4, 3, 36. But Jordanus's proofs are quite differently stated from those of Euclid or Leonardo. Both for themselves and for comparison with the Euclidean proofs which have come down to us, it will be interesting to reproduce propositions 13 and 17 of Jordanus.

"13. *Triangulo dato et puncto extra ipsum signato lineam per punctum transeuntem designare, que triangulum per equalia parciatur*" [11, 15-16].



"Let  $abc$  be the triangle and  $d$  the point outside but contained within the lines  $acf$ ,  $hbl$ , which are lines dividing the triangle equally and produced. For if  $d$  be taken in any such place, draw  $dg$  parallel to  $ca$  meeting  $cb$  produced in  $g$ . Join  $cd$  and find  $mn$  such that

$$\angle dg : \angle acd (= \frac{1}{2} \angle abc) = cg : mn.$$

\*Edited with Introduction by MAX CURTZE, *Mitteilungen des Gottinger Instituts für die Geschichte der Mathematik, Physik und Astronomie*, Vol. 1, 1883, pp. 1-11. The text of the edition is based on the edition of Heiberg, *Euclidis Elementa*, Vol. 1, 1883, pp. 1-11. One phase of his machinery has been referred to by ENESTROM, *Math. Ann.*, Vol. 1, 1883, pp. 1-11.

Then divide  $eg$  in  $k$  such that

$$gk : kc = kc : mn.$$

Produce  $dk$  to meet  $ca$  in  $p$ . Then I say that  $dp$  divides the triangle  $abc$  into equal parts.

For, since the triangle  $ckp$  is similar to the triangle  $kdg$ , by 4 of sixth<sup>56</sup> and parallel lines and 15 of first and definitions of similar areas,

$$\triangle ckp : \triangle kdg = mn : kg$$

by corollary to 17 of sixth<sup>57</sup>. But

$$\triangle kdg : \triangle cdg = kg : cg.$$

Therefore, by equal proportions,

$$\triangle ckp : \triangle cdg = mn : cg.$$

$$\therefore \triangle ckp : \triangle cdg = \triangle aec : \triangle cdg.$$

And

$$\triangle ckp = \triangle aec (= \frac{1}{2} \triangle abc)$$

by 9 of fifth, and this is the proposition.

And by the same process of deduction we may be led to an absurdity, namely, that all may equal a part if the point  $k$  be otherwise than between  $e$  and  $b$  or the point  $p$  be otherwise than between  $k$  and  $a$ ; the part cut off must always be either all or part of the triangle  $aec$ .

"17. *Puncto infra propositum trigonum dato lineam per ipsum deducere, quæ triangulum secet per equalia*" [pp. 17-18].

"Let  $abc$  be the triangle and  $d$  the point inside and contained within the part between  $ag$  and  $bc$  which divide two sides and triangle into equal parts. Through  $d$  draw  $fdh$  parallel to  $ac$  and draw  $db$ . Then by 12 of this book<sup>58</sup> draw  $mn$  such that

$$bf : mn = \triangle bdf : \triangle bcc (= \frac{1}{2} \triangle abc).$$

<sup>56</sup> That is, Euclid's *Elements* VI. 4.

<sup>57</sup> I do not know the MS. of Euclid here referred to; but manifestly it is the Porism of *Elements* VI. 19 which is quoted: "If three straight lines be proportional, then as the first is to the third, so is the figure described on the first to that which is similar and similarly described on the second."

<sup>58</sup> That is, *De Triangulis*, Book 2, Prop. 12: "Data recta linea aliam rectam invenire, ad quam se habeat prior sicut quilibet datus triangulus ad quemlibet datum triangulum" [p. 15].



Add then to the line  $cf$ , from  $f$ , a line  $fz$ , by 5 of this book<sup>100</sup>, such that

$$fz : cf :: mn : ac$$

and  $fz$  will be less than  $fb$  by the first part of the premise.

Supposition with regard to  $d$ ?]

Join  $zd$  and produce it to meet  $ac$  in  $k$ ; then I say that the line  $zdk$  divides the triangle  $abc$  into equal parts. For

$$\triangle bdf : \triangle zdf = bf : zf$$

by 1 of sixth.

$$\text{But } \triangle zdf : \triangle zkc = zf : mn$$

by corollary to 17 of sixth<sup>101</sup> and similar triangles.

Therefore by 1 and by equal proportions

$$\triangle bdf : \triangle zkc = bf : mn.$$

$$\text{But } \triangle bdf : \triangle bec = bf : mn.$$

Therefore by the second part of 9 of fifth

$$zkc = \triangle bec = \frac{1}{3} \triangle abc." \quad \text{Q. E. F.}$$

Proposition 18 of Jordanus is devoted to finding the centre of gravity of a triangle<sup>102</sup> and it is stated in the form of a problem on divisions. In Leonardo this problem is treated<sup>103</sup> by showing that the medians of a triangle are concurrent; but in Jordanus (as in Heron<sup>104</sup>) the question discussed is, "to find a point in a triangle such that when it is joined to the angular points, the triangle will be divided into three equal parts" (p. 18).

A much later work, *Summa de Arithmetica Geometria Proportioni et Proportionalita...* by Luca Paciuolo (b. about 1445) was published at Venice in 1494<sup>105</sup>. In the geometrical section (the second, and separately paged) of the work, pages 35 verso-43 verso, problems on divisions of figures are solved, and in this connection the author acknowledges great debt to Leonardo's work. Although the treatment is not as

<sup>100</sup> "Duabus lineis propositis, quarum una sit minor quarta alterius uel equalis, minori talem lineam adiungere, ut, que adiecte ad compositam, eadem sit composita ad reliquam propositarum proportio" [p. 12].

<sup>101</sup> Archimedes proved *Works of Archimedes*, Heath ed., 1897, p. 201; *Opera omnia* iterum edidit J. L. Heiberg, II, 150-159, 1913. In Propositions 13-14, Book 1 of "On the Equilibrium of Planes" that the centre of gravity of any triangle is at the intersection of the lines drawn from any two angles to the middle points of the opposite sides respectively.

<sup>102</sup> A new edition appeared at Toscolano in 1523, and in the section which we are discussing there does not appear to be any material change.



of the Dec book *On Divisions* was Muḥ. b. Muḥ. el-Baḡḡalāh who wrote at Cairo a table of sines for every minute. "A little later", however, Suter discovered facts which led him to believe that the true author was Abū Muhammed b. 'Abd el-Baḡḡ el-Baḡḡādī (d. 1141 at the age of over 70 years) to whom an excellent commentary on Book x of the *Elements* has been ascribed. Of a MS. by this author Gherard of Cremona (1114-1187) may well have been a translator.

Euclid's book *On Divisions* was undoubtedly the ultimate basis of all Arabian works on the same subject. We have record of two or three other treatises

1. Ṭābit b. Qorra (826-901) translated parts of the works of Archimedes and Apollonius, revised Ishāq's translation of Euclid's *Elements* and *Data* and also revised the work *On Divisions of Figures* translated by an anonymous writer.

2. Abū Muḥ. el-Hasan b. 'Obeidallāh b. Soleiman b. Waḥb (d. 901) was a distinguished geometer who wrote "A Commentary on the difficult parts of the work of Euclid" and "The Book on Proportion." Suter thinks<sup>7</sup> that another reading is possible in connection with the second title, and that it may refer to Euclid's work *On Divisions*.

3. Abū Wafā el-Būzḡāmī (940-997) one of the greatest of Arabian mathematicians and astronomers spent his later life in Bagdad, and is the author of a course of Lectures on geometrical constructions. Chapters VII-IX of the Persian form of this treatise which has come down to us in roundabout fashion were entitled: "On the division of triangles," "On the division of quadrilaterals," "On the division of circles" respectively. Chapter VII and the beginning of Chapter VIII are, however, missing from the Bibliothèque nationale Persian MS., which has been described by Woepeke<sup>8</sup>. This MS., which gives constructions without demonstrations, was made from an Arabian text, by one Abū Ishāq b. 'Abdallāh with

<sup>7</sup> H. Suter, *Die Mathematiker und ihre Werke*, Leipzig, 1902, p. 127.  
<sup>8</sup> H. Suter, *Die Mathematiker*, p. 127.

<sup>8</sup> H. Suter, *Die Mathematiker*, p. 127.  
<sup>9</sup> F. W. Rieu, *Les livres de géométrie d'Abū Wafā el-Būzḡāmī*, Paris, 1853, pp. 1-10.  
<sup>10</sup> F. W. Rieu, *Les livres de géométrie d'Abū Wafā el-Būzḡāmī*, Paris, 1853, pp. 1-10.  
<sup>11</sup> F. W. Rieu, *Les livres de géométrie d'Abū Wafā el-Būzḡāmī*, Paris, 1853, pp. 1-10.

the assistance of four pupils and the aid of another translation. The Arabian text was an abridgment of Abū'l Wefā's lectures prepared by a gifted disciple.

The three propositions of Chapter IX<sup>79</sup> are practically identical with Euclid (Woepeke) 28, 29. In Chapter VIII<sup>80</sup> there are 24 propositions. About a score are given, in substance, by both Leonardo and Euclid.

In conclusion, it may be remarked that in Chapter XII of Abū'l Wefā's work are 9 propositions, with various solutions, for dividing the surface of a sphere into equiangular and equilateral triangles, quadrilaterals, pentagons and hexagons.

20. *Practical applications of the problems On Divisions of Figures; the μετρικά of Heron of Alexandria.*—The popularity of the problems of Euclid's book *On Divisions* among Arabians, as well as later in Europe, was no doubt largely due to the possible practical application of the problems in the division of parcels of land of various shapes, the areas of which, according to the Rhind papyrus, were already discussed in empirical fashion about 1800 B.C. In the first century before Christ<sup>81</sup> we find that Heron of Alexandria dealt with the division of surfaces and solids in the third book of his *Surveying* (μετρικά)<sup>82</sup>. Although the enunciations of the propositions in this book are, as a whole, similar<sup>83</sup> to those

<sup>79</sup> Woepeke, *ibid.*, pp. 340-341; reprint, pp. 70-71.

<sup>80</sup> Woepeke, *ibid.*, pp. 338-340; reprint, pp. 67-69.

<sup>81</sup> It is not certain, but recent research appears to place Heron's life at about 150 A.D. Cf. H. V. THIRL, *Thirteen Books of Euclid's Elements*, I, 21, or perhaps better still, Article "Heron 2" by K. TATHE in Pauly-Wissowa's *Real-Encyclopädie der class. Altertumswissenschaften*, VIII, Stuttgart, 1909, especially columns 160-1600.

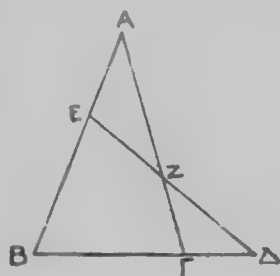
<sup>82</sup> *Heroni Alexandrini opera quae supersunt omnia*, Vol. III, *Libri III*, ed. Hermannus Schoene, Lipsiae, MCMIII. The 1<sup>st</sup> book, pp. 140-185. Cf. CANTOR, *Vorlesungen*, III, 1, 380-382.

<sup>83</sup> Only two are exactly the same: II III = Euclid 30, VII = Euclid 32, the problem considered in X is practically Euclid 27, Art. 48, while XVIII is closely related to Euclid 29, Art. 50<sup>th</sup>. In XIX Heron finds in a triangle a point such that the lines joined to the angular points, the triangle will be divided into three equal parts. The divisions of solids of which Heron treats are of a sphere (XXIII) and the division in a given ratio, by a plane parallel to the base, of a Pyramid (XX) and of a Cone (XXI). For proof of Proposition XXIII: *To cut a sphere by a plane so that the volumes of the segments are to one another in a given ratio*, Heron refers to Proposition 4, Book II of "On the Sphere and Cylinder" of Archimedes, the third proposition in the same book of the Archimedean work is Heron XVII: *To cut a given sphere by a plane so that the surfaces of the segments may have to one another a given ratio*. *Works of Archimedes*, Heath ed., 1897, pp. 61-65; *Opera omnia* iterum edita J. L. Heiberg, I, 184-195, 1010.

Propositions II and VII are also given in Heron's *περὶ διοπτρας* Schoene's

in Euclid's book *On Divisions*, Heron's discussion consists almost entirely of "analyses" and approximations. For example, 11: "To divide a triangle in a given ratio by a line drawn parallel to the base"—while Euclid gives the general construction, Heron considers that the sides of the given triangle have certain known numerical lengths and thence finds the approximate distance of the angular points of the triangle to the points in the sides where the required line parallel to the base intersects them, because, as he expressly states, in a field with uneven surface it is difficult to draw a line parallel to another. Most of the problems are discussed with a variety of numbers although theoretical analysis sometimes enters. Take as an example Proposition x<sup>4</sup>: "*To divide a triangle in a given ratio by a line drawn from a point in a side produced.*"

"Suppose the construction made. Then the ratio of triangle  $A EZ$  to quadrilateral  $Z E B \Gamma$  is known; also the ratio of the triangle  $A B \Gamma$  to the triangle  $A Z E$ . But the triangle  $A B \Gamma$  is known, therefore so is the triangle  $A Z E$ . Now  $\Delta$  is given. Through a known point  $\Delta$  there is therefore drawn a line which, with two lines  $A B$  and  $A \Gamma$  intersecting in  $A$ , encloses a known area.



Therefore the points  $E$  and  $Z$  are given. This is shown in the second book of *On Cutting off a Space*. Hence the required proof.

If the point  $\Delta$  be not on  $B \Gamma$  but anywhere this will make no difference."

21. *Connection between Euclid's book On Divisions, Apollonius's treatise On Cutting off a Space and a Pappus-lemma to Euclid's book of Porisms.*—Although the name of the author of the above-mentioned work is not given by Heron, the reference is clearly to Apollonius's lost work. According to Pappus it consisted of two books which contained 124 propositions treating of the various cases of the

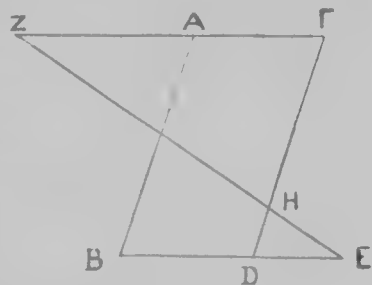
edition, pp. 278-281). Cf. "Extraits des Manuscrits relatifs à la géométrie grecque," par A. J. C. Vincent. *Notices et extraits des Manuscrits de la bibliothèque impériale*, Paris, 1858, XIX, pp. 157, 283, 285.

<sup>4</sup> HERON, *idem*, p. 160f.

following problem: *Given two coplanar straight lines  $A_1P_1$ ,  $B_1P_1$ , on which  $A_1$  and  $B_1$  are fixed points; it is required to draw through a fixed point  $\Delta$  of the plane, a transversal  $\Delta ZE$  forming on  $A_1P_1$ ,  $B_1P_1$  the two segments  $A_1Z$ ,  $B_1E$  such that  $A_1Z \cdot B_1E$  is equal to a given rectangle.*

Given a construction for the particular case when  $A_1P_1$ ,  $B_1P_1$  meet in  $A$ , and when  $A_1$  and  $B_1$  coincide with  $A$ —Heron's reasoning becomes clear. The solution of this particular case is practically equivalent to the solution of Euclid's Proposition 19 or 20 or 26 or 27. References to restorations of Apollonius's work are given in note 111.

To complete the list of references to writers before 1500, who have treated of Euclid's problems here under discussion, I should not fail to mention the last of the 38 lemmas which Pappus gives as useful in connection with the 171 theorems of Euclid's lost book of *Porisms*: *Through a given point  $E$  in  $BD$  produced to draw a line cutting the parallelogram  $AD$  such that the triangle  $Z\Gamma H$  is equal to the parallelogram  $AD$ .*



After "Analysis" Pappus has the following "Synthesis. Given the parallelogram  $AD$  and the point  $E$ . Through  $E$  draw the line  $EZ$  such that the rectangle  $\Gamma Z \cdot \Gamma H$  equals twice the rectangle  $AF \cdot \Gamma D$ . Then according to the above analysis [which contains a reference to an earlier lemma discussed a little later\* in this book] the triangle  $Z\Gamma H$  equals the parallelogram  $AD$ . Hence  $EZ$  satisfies the problem and is the only line to do so."

The tacit assumption here made, that the equivalent of a proposition of Euclid's book *On Divisions (of Figures)* was well known, is noteworthy.

\* Pappus ed. by Hultsch, Vol. 2, Berlin, 1877, pp. 917-919. In Chasles's restoration of Euclid's *Porisms*, this lemma is used in connection with "Porism CLXXX. Given two lines  $SA$ ,  $SA'$ , a point  $P$  and a space  $v$ : points  $I$  and  $J'$  can be found in a line with  $P$  and such that if one take on  $SA$ ,  $SA'$  two points  $m$ ,  $m'$ , bound by the equation  $Im \cdot J'm' = v$ , the line  $mm'$  will pass through a given point." *Les trois livres de Porismes d'Euclide*, Paris, 1860, p. 284. See also the restoration by R. Simson, pp. 527-530 of "De porismatibus tractatus," *Opera quaedam reliqua...* Glasguae, MDCCXXVI.

### III.

*"The Treatise of Euclid on the Division (of plane Figures)."*

# PROPOSITION 1.

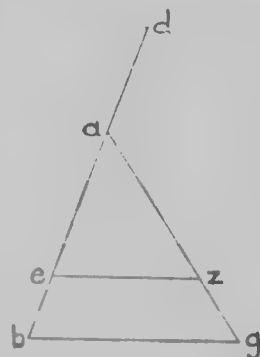
22. *To divide a given triangle into equal parts by a line parallel to its base.* [Leonardo 5, p. 119, ll. 7-9.]

Let  $abg$  be the given triangle which it is required to bisect by a line parallel to  $bg$ . Produce  $ba$  to  $d$  till  $ba = ad$ . Then in  $ba$  find a point  $e$  such that

$$ba : ae = ac : ad.$$

Through  $e$  draw  $ez$  parallel to  $bg$ ; then the triangle  $abg$  is divided by the line  $ez$  into two equal parts, of which one is the triangle  $aez$ , and the other the quadrilateral  $ebgz$ .

Leonardo then gives three proofs, but as the first and second are practically equivalent, I shall only indicate the second and third.



I. When three lines are proportional, as the first is to the third so is a figure on the first to the similar and similarly situated figure described on the second [vi. 19, "Porism" <sup>1</sup>].

$\therefore ba : ad =$  figure on  $ba$  : similar and similarly situated figure on  $ac$ .

Hence  $ba : ad = \triangle abg : \triangle aez$ .

$$= 2 : 1.$$

$$\therefore \triangle abg = 2 \triangle aez.$$

II.  $ba : ac = ac : ad$ .

$$\therefore ba : ad = ac^2.$$

<sup>1</sup> Literally, the original runs, according to Woepcke, "We propose to ourselves to demonstrate how to divide, etc." I have added all footnotes except those attributed to Woepcke.

<sup>2</sup> Throughout the restoration I have added occasional references of this kind to Heath's edition of Euclid's *Elements*; vi. 19 refers to Proposition 10 of Book VI.



## PROBLEM 2.

23. To divide a given triangle into three equal parts by two lines parallel to its base. Leonardo 14, p. 122 [1, 8, 1]

Now  $ba = 3ad$ ;  $\therefore \triangle abg = 3\triangle a\tau t$ .

$$\therefore a\tau t = \frac{1}{3}\triangle abg.$$

Again,  $ba : ia = ia : ac$ ;

$\therefore ba : ac = \triangle$  on  $ca$  : similar and similarly situated  $\triangle$  on  $ai$ .

But triangles  $aik$ ,  $abg$  are similar and similarly described on  $ai$  and  $ab$ ; and

$$ca : ab = 2 : 3,$$

$$\therefore aik = \frac{2}{3}\triangle abg.$$

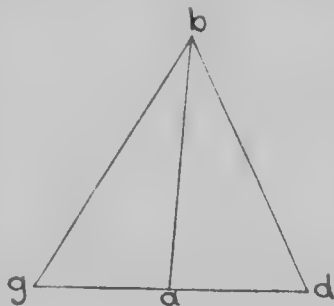
And since  $\triangle a\tau t = \frac{1}{3}\triangle abg$ , there remains the quadrilateral  $ikt = \frac{1}{3}\triangle abg$ . We see that the quadrilateral  $ibgk$  will be the other third part; hence the triangle  $abg$  has been divided into three equal parts; "quod oportebat facere."

Leonardo continues: "Et sic per demonstratos modos omnia genera trigonorum possunt diuidi in quatuor partes uel plures." Cf. note 45.

### PROPOSITION 3.

24. "To divide a given triangle into two equal parts by a line drawn from a given point situated on one of the sides of the triangle." [Leonardo I, 2, p. 110, l. 31; p. 111, ll. 41-43.]

Given the triangle  $bgd$ ; if  $a$  be the middle point of  $gd$  the line  $ba$  will divide the triangle as required; either because the triangles are on equal bases and of the same altitude [I. 38; Leonardo I], or because



$$\triangle bgd : \triangle bad = bd . dg : bd . da^*.$$

Whence

$$\triangle bgd = 2\triangle bad.$$

But if the given point be not the middle point of any side, let  $abg$  be the triangle and  $d$  the given point nearer to  $b$  than to  $g$ . Bisect  $bg$  at  $e$  and draw  $ad$ ,  $ac$ . Through  $e$  draw  $ez$  parallel to  $da$ ; join  $dz$ . Then the triangle  $abg$  is bisected by  $dz$ .

*Proof:* Since

$$ad = ce, \quad \angle adz = \angle ade.$$

To each add  $\angle abd$ . Then

$$\text{quadr. } abd = \angle abd + \angle ade.$$

$$= \angle e.$$

But

$$\angle e = \angle zdg;$$

$$\therefore \text{quadr. } abd = \angle zdg;$$

and the triangle  $zdg$  is the other half of the triangle  $abg$ . Therefore the triangle  $abg$  is divided into two equal parts by the line  $dz$  drawn from the point  $d$ .

Quod oportebat facere.

FIG. 11. A triangle bisected by a line.

#### PROPOSITION 4.

25. "To divide a given trapezium" into two equal parts by a line parallel to its base." [Leonardo 23, p. 125, ll. 37-38.]

Let  $abgd$  be the given trapezium with parallel sides  $ad$ ,  $bg$ ,  $ad$  being the lesser. It is required to bisect the trapezium by a line parallel to the base  $bg$ . Let  $gd$ ,  $ba$ , produced, meet in a point  $e$ . Determine  $z$  such that

$$ez = \frac{1}{2}(eb + ea).$$

Through  $z$  draw  $zi$  parallel to  $gb$ . I say that the trapezium  $abgd$  is divided into two equal parts by the line  $zi$  parallel to the base  $bg$ .

\* Here, and in what follows, the word *quadr.* is used to refer to a quadrilateral two of whose sides are parallel.

\* The point  $z$  is a vertex of the trapeziums which two make use of in 27.



Divide  $tz$  into three equal parts  $tk$ ,  $kl$ ,  $lz$ .

Find  $m$  and  $n$  in  $bc$  such that

$$em : ek = tk : tl,$$

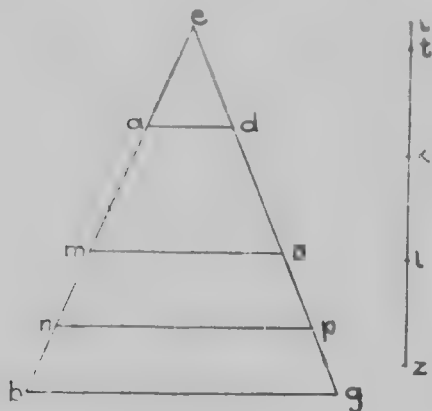
and

$$en : eb = il : iz.$$

Through  $m$  and  $n$  draw  $mo$ ,  $np$  parallel to the base  $bg$ . Then I say that the quadrilateral  $ag$  is divided into three equal parts:  $ao$ ,  $mp$ ,  $ng$ .

*Proof:* For  $eb : ea = \angle cbg : \angle cad$ . [VI. 19]

$$\therefore iz : it = \angle cbg : \angle cad. \dots\dots\dots[1]$$



But  $zi : tk = eb : em$ ,

$$\therefore zi : tk = \angle cbg : \angle emg. \dots\dots\dots[2]$$

So also  $zi : il = \angle cbg : \angle enp. \dots\dots\dots[3]$

Whence  $it : tk = \angle cad : \text{quadl. } ao$ ,<sup>9</sup>

and therefore  $tk : kl = \text{quadl. } ao : \text{quadl. } mp$ .<sup>10</sup>

<sup>9</sup> This may be obtained by combining [1] and [2] and applying V. 14, 16, 17.

<sup>10</sup> Relations [1], [2] and [3] may be employed, as in the preceding, to give,

$$it : kl = \angle cad : \text{quadl. } mp;$$

combining this with  $it : tk = \angle cad : \text{quadl. } ao$ , we get the required result.

$$tk : kl = \text{quadl. } ao : \text{quadl. } mp.$$

But  $tk = kl$ ,  $\therefore$  quadr.  $ao =$  quadr.  $mp$ .  
 So also  $kl : lz =$  quadr.  $mp : \text{quadr. } ng$ .  
 and  $kl = lz$ ,  $\therefore$  quadr.  $mp =$  quadr.  $ng$ .

Therefore the quadrilateral is divided into equal quadrilaterals  $ao$ ,  $mp$ ,  $ng$ ;  
 "ut prediximus.

Then follows a numerical example.

### PROPOSITION 6.

27. "To divide a parallelogram into two equal parts by a straight line drawn from a given point situated on one of the sides of the parallelogram." [Leonardo 16, p. 123, ll. 30-31.]

Let  $abcd$  be the parallelogram and  $i$  any point in the side  $ad$ . Bisect  $ad$  in  $f$  and  $bc$  in  $e$ . Join  $fe$ . Then the parallelogram  $ac$  is divided into equal parallelograms  $ac$ ,  $fc$  on equal bases.

Cut off  $ch = fi$ . Join  $hi$ . Then this is the line required.

Leonardo gives two proofs.

I. Let  $hi$  meet  $fe$  in  $k$ . Then  $\angle s fki$ ,  $hke$  are equal; add to each the pentagon  $kfabh$ , etc.

II. Since  $ac$ ,  $fc$  are  $\square$ 's,  $af = be$  and  $fd = ec$ . But

$$fd = \frac{1}{2}ad.$$

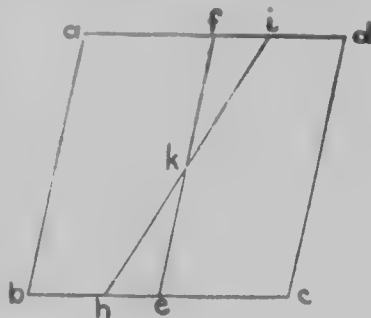
$$\therefore fd = af = ac.$$

And since  $fi = he$ ,  $ai = ch$ .

So also  $di = bh$ , and  $hi$  is common.

$$\therefore \text{quadr. } iabh = \text{quadr. } ihcd.$$

The first rather than the second proof is Euclidean. There is no proposition of the *Elements* with regard to the equality of quadrilaterals whose sides and angles, taken in the same order, are equal. Of course the result is readily deduced from I. 4, if we make certain suppositions with regard to order. Cf. the proof of Prop. 10.



Similarly if the given point were between  $a$  and  $f$  [etc. or on any other side]. And thus a parallelogram can be divided into two equal parts by a straight line drawn from a given point at random to the opposite side.

PROPOSITION 7.

28. *Quodlibet parallelogramum in quatuor partes aequales dividitur per a straight line drawn from a given point situated at random to the opposite side.* [Leonardo 26 (the case where the fraction is one-third) pp. 14-15].

Let  $abcd$  be the given parallelogram. Suppose it be required to cut off a third of this parallelogram by a straight line drawn from  $i$ , in the line  $ad$ .

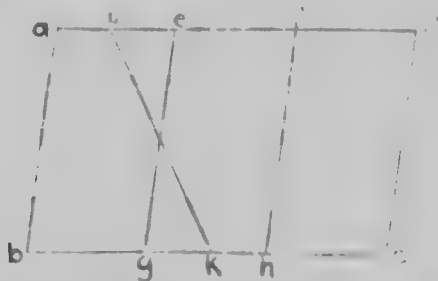


Fig. 1. The bisecting of a parallelogram into two equal parts.

Trisect  $ad$  in  $e$  and  $f$  and through  $e, f$  draw  $eg, fh$  parallel to  $dc$ ; [then these lines trisect the  $\square$ ]. If the point  $i$  be in the line  $ad$  at either  $e$  or  $f$ , then the problem is solved. But if it be between  $a$  and  $e$ , draw  $ik$  to bisect the  $\square ah$  (Prop. 6), etc. Similarly if  $i$  were between  $e$  and  $f$ , or between  $f$  and  $d$ .

After finishing these cases Leonardo concludes:

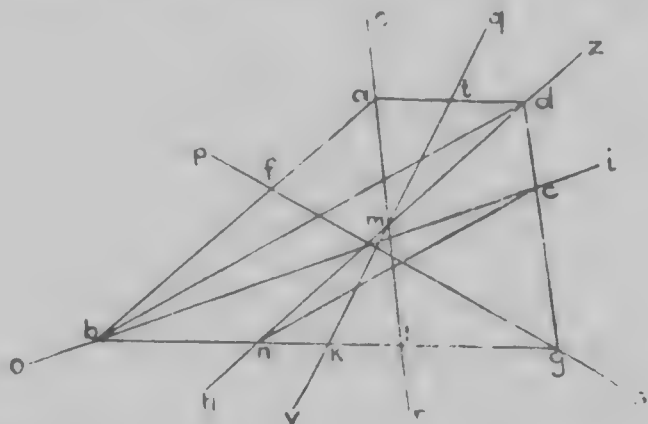
"eodem modo potest omnem parallelogramum diuidi in quatuor uel plures partes equales".

The construction in this proposition is limited to the case where "a certain fraction" of the parallelogram is required. But Leonardo's method of construction were  $m:n$  (the ratio of the lengths of two given lines), would be extended in a very similar way. Divide  $ad$  into  $n$  equal parts, so that  $ae = \frac{1}{n}ad$ ,  $af = \frac{m-1}{n}ad$ . In  $ad$  cut off  $ef = ae$  and through  $f$  draw  $fh$  parallel to  $ab$ . Then, as before, the problem is reduced to Proposition 6.

If the point  $e$  should fall at  $i$  or in the interval  $ai$  the part cut off from the parallelogram by the required line would be in the form of a triangle which might be determined by Prop. 6.

## 8.

1. 27, 11, 2, 3

are inserted in  $t$  and  $k$  respectively. As  $\mathcal{L}(\mathcal{A}) \subseteq \mathcal{L}(\mathcal{B})$ , there is a  $\beta$  in  $\mathcal{B}$  such that

between the same parallels, that  $AC$  bisects the trapezium

This is Leonardo 24, p. 12<sup>v</sup>, l. 31

equal to  $AC$ . Join  $A'$ , meeting  $AC$  in  $B'$ , then the quadrilateral

to  $td$ , and  $dn$  divides the quadrilateral into two equal parts.

[Were the given point anywhere between  $a$  and  $l$  the other end of the bisecting line would be between  $k$  and  $l$ . Similarly if the given point were between  $l$  and  $d$ , the corresponding point would be between  $k$  and  $n$ .]

Although not observed by Favaro, Leonardo now considers:

If the given point be in the side  $bg$ ; either  $l$ , or  $n$ , or a point between  $l$  and  $n$ , then the above construction is at once applicable.

Suppose, however, that the given point were at  $b$  or in the segment  $bn$ , at  $g$  or in the segment  $lg$ . First consider the given point at  $b$ . Join  $bd$  and through  $n$  draw  $nc$  parallel to  $bd$  to meet  $gd$  in  $c$ . Join  $bc$ . Then  $bc$  bisects the trapezium. For  $abnd$  is half of the trapezium  $ag$ , and the triangle  $bnd$  equals the triangle  $bdc$  etc.

Similarly from a given point between  $b$  and  $n$ , a line could be drawn meeting  $gd$  between  $c$  and  $d$ , and dividing the quadrilateral into two equal parts.

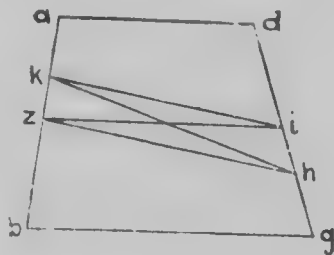
So also from  $g$  a line  $gf$  could be drawn [etc.]; and similarly for a given point between  $g$  and  $l$ .

Leonardo then concludes (p. 127, ll. 37-40):

"Jam ostensum est quomodo in duo equa quadrilatera duorum equidistantium laterum diuidi debeant á linea protracta ab omni dato puncto super lineas equidistantes ipsius; nunc uero ostendamus quomodo diuidantur á linea egrediente á dato puncto super reliqua latera."

This is overlooked by Favaro, though implied in his 27 [Leonardo, p. 129, l. 4]. I may add Leonardo's discussion of the above proposition although it does not seem to be called for by Euclid.

Let the point be in the side  $gd$ . For  $g$  or  $c$  or  $d$  or any point between  $c$  and  $d$  the above constructions clearly suffice. Let us, then, now consider the given point  $h$  as between  $c$  and  $g$ . Draw the line  $iz$  parallel to  $gb$  to bisect the trapezium (Prop. 4). Suppose  $h$  were between  $g$  and  $i$ . Join  $zh$ . Through  $i$  draw  $ik$  parallel to  $hz$ , and meeting  $ab$  in  $k$ .



(The lettering of the original figure is somewhat changed.)



Join  $tk$ . Then by joining  $bt$  and  $gt$  [it is easily seen by VI. 1 and V. 12, that the trapezium  $ag$  is divided by  $tk$  in the ratio  $c : d$ ].

If the given point be at  $a$  or  $d$ , make  $kl = at$  and  $gn = bl$ . Join  $al$ ,  $dn$ . [Adding the quadrilateral  $ak$  to the congruent triangles with equal sides  $at$ ,  $kl$ , we find  $al$  divides the trapezium in the required ratio. Then from VI. 1,  $dn$  does the same.

As in Proposition 8, for any point  $t'$  between  $a$  and  $t$ , or  $t$  and  $d$ , we have a corresponding point  $k'$  between  $l$  and  $k$  or  $n$  and  $k$ , such that the line  $t'k'$  divides the trapezium in the given ratio.

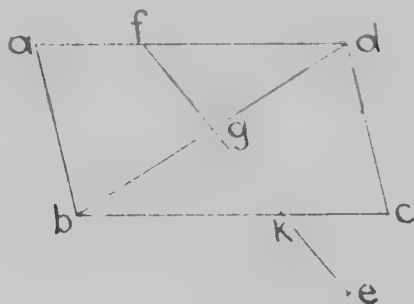
If the given point be in  $bg$  at  $l$  or  $n$  or between  $l$  and  $n$ , the above reasoning suffices.

Suppose however that the given point were at  $b$ . Join  $bd$ . Through  $n$  draw  $nc$  parallel to  $bd$ . Join  $bc$ . Then  $bc$  divides the trapezium in the required ratio. Similarly for the point  $g$  and for any point between  $b$  and  $n$ , or between  $g$  and  $l$ .

Some of the parts which I have filled in above are covered by the general final statement: "*nec non et diuidemus ipsum quadrilaterum ab omni puncto dato super aliquod laterum ipsius.....*" (Page 134, ll. 10-11. Compare Proposition 13.)

#### PROPOSITION 10.

31. "*To divide a parallelogram into two equal parts by a straight line drawn from a given point outside the parallelogram.*" [Leonardo 18, p. 124, ll. 5-7.]



Let  $abcd$  be the given parallelogram and  $e$  the point outside. Join  $bd$  and bisect it in  $g$ . Join  $eg$  meeting  $bc$  in  $k$

and produce it to meet  $ad$  in  $f$ . Then the parallelogram has been divided into two equal parts by the line drawn through  $e$ , as may be proved by superposition; and one half is the quadrilateral  $fabb$ , the other, the quadrilateral  $fkcd$ <sup>99</sup>.

### PROPOSITION 11.

32. "*To cut off a certain fraction from a parallelogram by a straight line drawn from a given point outside of the parallelogram.*"

This proposition is not explicitly formulated by Leonardo; but the general method he would have employed seems obvious from what has gone before.

Suppose it were required to cut off one-third of the given parallelogram  $ac$  by a line drawn through a point  $e$  outside of the parallelogram. Then by the method of Proposition 7, form a parallelogram two-thirds of  $ac$ . There are four such parallelograms with centres  $g_1, g_2, g_3, g_4$ . Lines  $l_1, l_2, l_3, l_4$  through each one of these points and  $e$  will bisect a parallelogram (Proposition 10).

There are several cases to consider with regard to the position of  $e$  but it may be readily shown that, in one case at least, there is a line  $l_i (i = 1, 2, 3, 4)$ , which will cut off a third of the parallelogram  $ac$ .

Similarly for one-fourth, one-fifth, or any other fraction such as  $m:n$  which represents the ratio of lengths of given lines.

### PROPOSITION 12.

33. "*To divide a given trapezium into two equal parts by a straight line drawn from a point which is not situated on the longer side of the trapezium. It is necessary that the point be situated beyond the points of concurrence of the two sides of the trapezium.*" [Leonardo 28, p. 129, ll. 2-4, and another, unnumbered<sup>100</sup>.]

### PROPOSITION 13.

34. "*To cut off a certain fraction from a (parallel-) trapezium by a straight line which passes through a given point lying inside or outside the trapezium but so that a straight line can be drawn through it cutting both the parallel*

<sup>99</sup> The proof also follows from the equality of the triangles  $fcd, bck$ , by 1. 26 and of the triangles  $abd, bdc$  by 1. 4. This problem is possible for all positions of the point  $e$ .

<sup>100</sup> As Leonardo 28 Favaro gives, "Qualiter quadrilatera duorum laterum equidistantium dividi debeant a dato puncto extra figuram" and entirely ignores the paragraph headed, "De diuisione eiusdem generis, qua quadrilaterorum per rectam transeuntem per punctum datum infra ipsum" [p. 131, ll. 13-14]

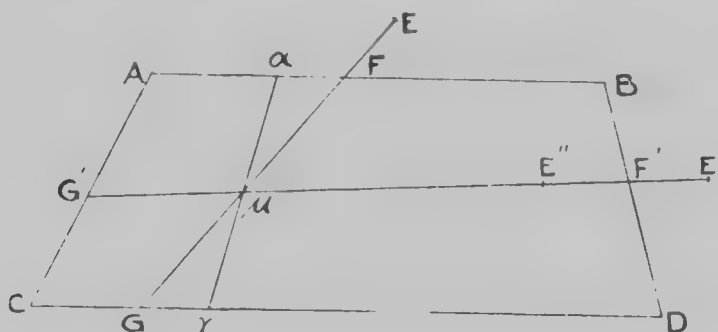
sides of the trapezium<sup>101</sup>." [Part of Leonardo 32<sup>102</sup>, p. 134, ll. 11-12.]

We first take up Leonardo's discussion of Proposition 12.

In the figure of Proposition 8, suppose  $al$  to be produced in the directions of the points  $e$  and  $r$ ;  $lk$  in the directions of  $q$  and  $v$ ,  $du$  of  $z$  and  $h$ ,  $cb$  of  $i$  and  $o$ ,  $gf$  of  $s$  and  $p$ . Then for [any such exterior points  $e, q, z, i, s, r, v, h, o, p$ , lines are drawn bisecting the trapezium].

If the given point,  $x$ , were anywhere in the section of the plane above  $ad$  and between  $ca$  and  $dz$ , the line joining  $x$  to  $m$  would [by the same reasoning as in Proposition 8] bisect the trapezium. Similarly for all points below  $nl$  and between

<sup>101</sup> The final clauses of Propositions 12 and 13, in Woepcke's rendering, are the same. I have given a literal translation in Proposition 12. Heath's translation and interpretation (after Woepcke) are given in 13. Concerning 12 and 13 Woepcke adds the following note: "Suppose it were required to cut off the  $n$ th part of the trapezium  $ABDC$ ; make  $Aa$  and  $Cy$  respectively equal to the  $n$ th parts



of  $AB$  and of  $CD$ ; then  $AayC$  will be the  $n$ th part of the trapezium, for  $ya$  produced will pass through the intersection of  $CA$ ,  $DB$  produced. Now to draw through a given point  $E$  the transversal which cuts off a certain fraction of the trapezium, join the middle point  $\mu$  of the segment  $ay$ , and the point  $E$ , by a line; this line  $EFG$  will be the transversal required to be drawn, since the triangle  $aF\mu$  equals the triangle  $\gamma G\mu$ .

"But when the given point is situated as  $E'$  or  $E''$  such that the transversal drawn through  $\mu$  no longer meets the two parallel sides but one of the parallel sides and one of the two other sides, or the other two sides; then the construction indicated is not valid since  $C\gamma\mu y$  is not equal to  $BF'\mu a$ . It appears that this is the idea which the text is intended to express. The 'points of concurrence' are the vertices where a parallel side and one of the two other sides intersect; and the expression 'beyond' refers to the movement of the transversal represented as turning about the point  $\mu$ ."

<sup>102</sup> "Quadrilaterum [trapezium] ab omni puncto dato super aliquod laterum positus, et etiam ab omni puncto dato infra, uel extra diuidere in aliqua data proportioni."

*hn* and *lr* [.....so also for all points within the triangles *amd*, *nml*].

This seems to be all that Euclid's Proposition 12 calls for. But just as Leonardo considers Proposition 8 for the general case with the given point anywhere on the perimeter of the trapezium, so here, he discusses the constructions for drawing a line from any point inside or outside of a trapezium to divide it into two equal parts.

Leonardo does not give any details of the discussion of Euclid's Proposition 13, but after presentation of the cases given in Proposition 9 concludes: "*et diuidemus ipsum quadrilaterum ab omni puncto dato super aliquod laterum ipsius, et etiam ab omni puncto dato infra uel extra*" [Leonardo 32, p. 134, ll. 10-12].

From Leonardo's discussion in Propositions 8, 9, 12, not only are the necessary steps for the construction of 13 (indicated in the Woepeke note above<sup>101</sup>) evident, but also those for the more general cases, not considered by Euclid, where restrictions are not imposed on the position of the given point.

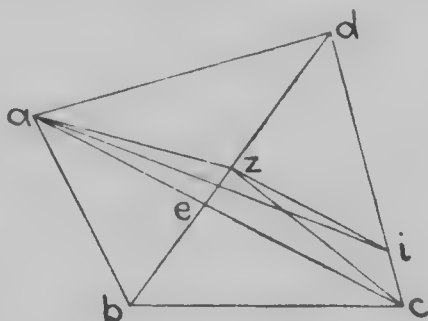
#### PROPOSITION 14.

35. "*To divide a given quadrilateral into two equal parts by a straight line drawn from a given vertex of the quadrilateral.*" [Leonardo 36, p. 138, ll. 10-11.]

Let *abcd* be the quadrilateral and *a* the given vertex. Draw the diagonal *bd*, meeting the diagonal *ac* in *e*. If *be*, *ed* are equal, [*ac* divides the quadrilateral as required].

If *be* be not equal to *ed*, make *bz* = *zd*.

Draw *zi* || *ac* to meet *dc* in *i*. Join *ai*. Then the quadrilateral *abcd* is divided as required by the line *ai*.



*Proof:* Join *az* and *zc*. Then the triangles *abz*, *azd* are respectively equal to the triangles *cbz*, *cdz*.

Therefore the quadrilateral  $abcz$  is one-half the quadrilateral  $abcd$ .

And since the triangles  $azc$ ,  $aic$  are on the same base and between the same parallels  $ac$ ,  $zi$ , they are equal.

To each add the triangle  $abc$ .

Then the quadrilateral  $abcz$  is equal to the quadrilateral  $abei$ . But the quadrilateral  $abcz$  is one-half of the quadrilateral  $abcd$ . Therefore  $abei$  is one-half of the quadrilateral  $abcd$ ;  
"ut oportet."

### PROPOSITION 15.

36. "To cut off a certain fraction from a given quadrilateral by a line drawn from a given vertex of the quadrilateral." [Leonardo 40, p. 140, ll. 36-37.]

Let the given fraction be as  $cz : zi$ , and let the quadrilateral be  $abcd$  and the given vertex  $d$ . Divide  $ac$  in  $t$  such that

$$at : tc = cz : zi.$$

If  $bd$  pass through  $t$  [then  $bd$  is the line required].

But if  $bd$  do not pass through  $t$  it will intersect either  $ct$  or  $ta$ ; let it intersect  $ct$ . Join  $bt$ ,  $td$ .

Then

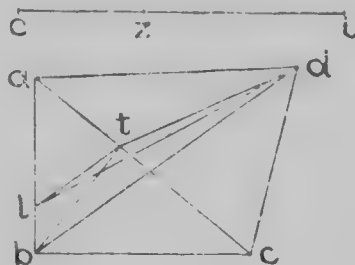
$$\text{quadr. } tbcd : \text{quadr. } tbad = ct : ta = cz : zi.$$

Draw  $tl$  parallel to the diagonal  $bd$ , and join  $dl$ . Then the quadrilaterals  $tbcd$ ,  $tlcd$  are equal and the construction has been made as required; for

$$ct : ta = cz : zi = \text{quadr. } tbcd : \triangle dal.$$

And if  $bd$  intersect  $ta$  a similar construction may be given to divide the given quadrilateral, by a line through  $d$ , into a quadrilateral and triangle in the required ratio].

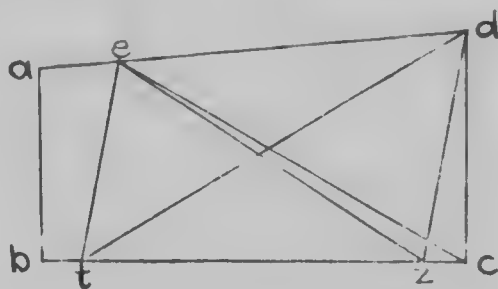
Leonardo then gives the construction for dividing a quadrilateral in a given ratio by a line drawn through a point which divides a side of the quadrilateral in the given ratio.



## PROPOSITION 16.

37. "To divide a given quadrilateral into two equal parts by a straight line drawn from a given point situated on one of the sides of the quadrilateral." [Leonardo 37, p. 138, ll. 28-29.]

Let  $abcd$  be the given quadrilateral,  $e$  the given point. Divide  $ac$  into two equal parts by the line  $dt$  [Prop. 14]. Join  $et$ . The line  $et$  either is, or is not, parallel to  $dc$ .



Two of Leonardo's figures are combined in one, here.)

If  $et$  be parallel to  $dc$ , join  $ec$ . Then the quadrilateral  $ac$  [is bisected by the line  $ec$ , etc.].

If  $et$  be not parallel to  $dc$ , draw  $dz \parallel et$ . Join  $ez$ . Then  $ac$  [is bisected by the line  $ez$ , etc.].

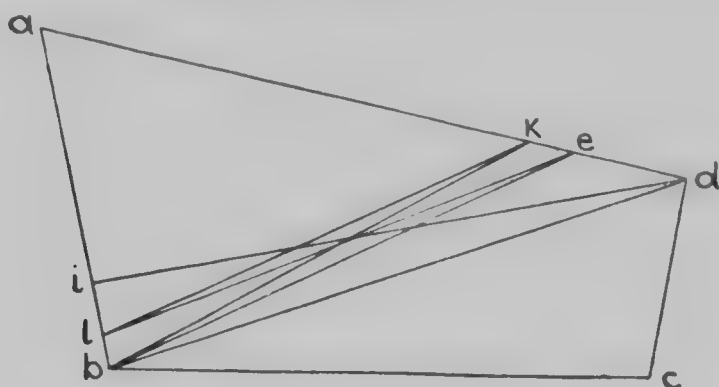
Leonardo does not consider the case of failure of this construction, namely when  $dz$  falls outside the quadrilateral. Suppose in such a case that the problem were solved by a line joining  $e$  to a point  $z'$  (not shown in the figure) on  $dc$ . Through  $t$ , draw  $tt' \parallel ad$ . Join  $et'$ . Then  $\angle et'd = \angle ctd = \angle edc$ . Whence  $\triangle et'e = \triangle z'e'e$ , or  $t'z' \parallel ee$ . Therefore from  $t'$ ,  $z'$  may be found and the solution in this case is also possible, indeed in more than one way, but it is not in Euclid's manner to consider this question.

Should the diagonal  $db$  bisect the quadrilateral  $ac$ , the discussion is similar to the above.

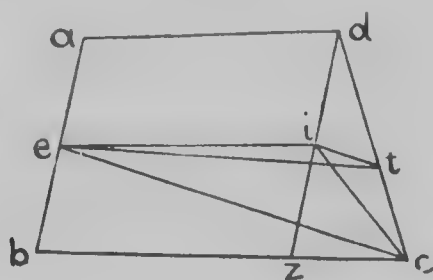
But if the line drawn from  $d$  to bisect the quadrilateral meet the side  $ab$  in  $i$ , draw  $bk$  bisecting the quadrilateral  $ac$ .

If  $k$  be not the given point, it will be between  $k$  and  $d$  or between  $k$  and  $a$ .

In the first case join  $be$  and through  $k$  draw  $k'l \parallel cb$ . Join  $cl$  [then  $cl$  is the required bisector].



If the point  $e$  be between  $a$  and  $k$  [a similar construction with the line through  $k$  parallel to  $bc$ , and meeting  $bc$  in  $m$ , leads to the solution by the line  $cm$ ].



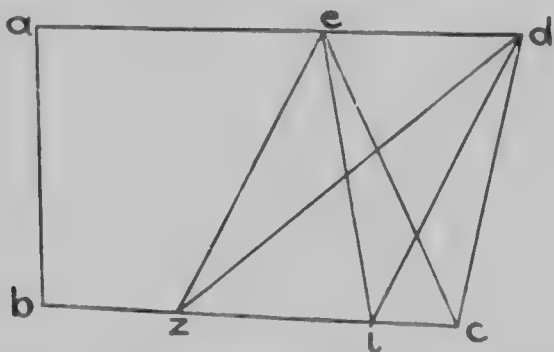
Were  $e$  at the middle of a side such as  $ab$ , draw  $dz \parallel ab$  and bisect  $dz$  in  $i$ . Join  $ei$ ,  $ci$  and  $ec$ . Through  $i$  draw  $it \parallel ec$ . Join  $et$ ; then  $et$  [bisects the quadrilateral  $ac$ , since  $\triangle itc = \triangle ite$ , etc.].

If  $dz$  were to fall outside the quadrilateral, draw from  $e$  the parallel to  $ba$ ; and so on

### PROPOSITION 17.

38. "To cut off a certain fraction from a quadrilateral by a straight line drawn from a given point situated on one of the sides of the quadrilateral." [Leonardo 39. p. 140, ll. 11-12.]

Let  $abcd$  be the given quadrilateral and suppose it be required to cut off one-third by a line drawn from the point  $c$  in the side  $ad$ .



Draw  $dz$  cutting off one-third of  $ac$  [Prop. 15].

Join  $ez$ ,  $ec$ .

If  $ez \parallel dc$ , then  $ecd$  [is the required part cut off, etc.].

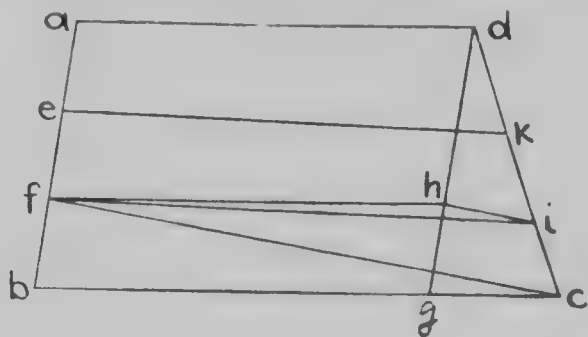
But if  $ez$  be not parallel to  $dc$ , draw  $di \parallel ez$  and join  $ei$ . [Then this is the line required, etc.]

The case when  $ei$  cuts  $dc$  is not taken up but it may be considered as in the last proposition.

So also to divide  $ac$  into any ratio: draw  $dz$  dividing it in that ratio (Prop. 15), and then proceed as above.

A particular case which Leonardo gives may be added.

Let  $ab$  be divided into three equal parts  $ae$ ,  $ef$ ,  $fb$ ; draw  $dg \parallel ab$  and cut off  $gh = \frac{1}{3}gd$ . Join  $fc$  and through  $h$  draw



$hi \parallel fc$ , meeting  $dc$  in  $i$ . Join  $fi$ ; and the quadrilateral  $fbci$  will be one-third of the quadrilateral  $ac$ . [As in latter part of Prop. 16.]



"After having done what was required, if some one ask, How is it possible to apply to the line  $AB$  a rectangle such



that the rectangle  $AE \cdot EB$  is equal to the rectangle  $AB \cdot AC$  and deficient by a square—we say that it is impossible, because  $AB$  is greater than  $BE$  and  $AC$  greater than  $AE$ , and consequently the rectangle  $BA \cdot AC$  greater than the rectangle  $AE \cdot EB$ . Then when one applies to the line  $AB$  a parallelogram equal to the rectangle  $AB \cdot AC$  the rectangle  $AZ \cdot ZB$  is....."

In this problem it is required to find in the given line  $AB$  a point  $Z$  such that

$$AB \cdot ZB = ZB^2 [- AZ \cdot ZB \text{ by II. 3; cf. X. 16 lemma}] - AB \cdot AC.$$

Find, by II. 14, the side,  $b$ , of a square equal in area to the rectangle  $AB \cdot AC$ ; then the problem is exactly equivalent to that of which a simple solution was given by Simson:

*Thus the given rectilinear figure must not be greater than the square described on the half of the straight line and similar to the given figure.*

The Proposition 18 of Euclid under consideration is a particular case of the problem and as the fragment of the text and Woepcke's note 104<sup>1</sup> contained in it, doubt may well be entertained as to whether Euclid gave a construction in his book *On Divisions*. The problem can be solved without the aid of Book VI of the *Elements* and by means of II. 5 and II. 14 only, as indicated in the text above.

The appropriation of the terms parabola *application*, hyperbola *exceeding*, and ellipse *falling short* to conic sections was first introduced by Apollonius as expressing in each case the fundamental properties of curves as stated by him. This fundamental property is the geometrical equivalent of the Cartesian equation referred to any diameter of the conic and the tangent at its extremity as—in general, oblique—axes. More particulars in this connection are given by Heath.

The terms "parabolic," "hyperbolic" and "elliptic," introduced by Klein for the three main divisions of Geometry, are appropriate to systems in which a straight angle equals, exceeds and falls short of the angle sum of any triangle. Cf. W. B. FRANKLAND, *The First Book of Euclid's Elements with a commentary based principally upon that of Proclus Diadochus*...Cambridge, 1905, p. 122.

<sup>104</sup> Woepcke here remarks: "Evidently if  $a$  denote the length of the line to which the rectangle is to be applied, Problem 18 is only possible when

$$AB \cdot AC < \left(\frac{1}{2} AB\right)^2$$

Then if  $a$  be taken as  $AB$  one of the two sides of the given rectangle, relative to the other side,  $AC < \frac{AB}{4}$ . It is probably the demonstration of this which was given in the missing portion of the text.

<sup>105</sup> If  $AB = a$ ,  $ZB = x$ ,  $AB \cdot AC = b^2$ , the problem is to find a geometric solution of the equation  $ax - x^2 = b^2$ . Offerdinger<sup>3a</sup> p. 15 seems to have quite missed the meaning of this problem. He thought, apparently, that it was equivalent to X. 16, lemma of the *Elements*.

<sup>106</sup> R. SIMSON, *Elements of Euclid*, ninth ed., Edinburgh, 1793, pp. 335-4







Then apply to  $ZB$  a rectangle equal to the rectangle  $ZB \cdot BE$  and deficient by a square. [Prop. 18.] Let the rectangle applied be the rectangle

$$BH \cdot HZ [(ZB - HZ) HZ = ZB \cdot BE].$$

Draw the line  $HD$  and produce it to  $T$ .

"On proceeding as above we may demonstrate that the triangle  $HTB$  is one-third of the triangle  $ABC$ ; and by means of an analogous construction to this we may divide the triangle in any ratio. But this is what it is required to do<sup>10</sup>."

### PROPOSITION 21.

42. "Given the four lines  $A, B, C, D$  and that the product of  $A$  and  $D$  is greater than the product of  $B$  and  $C$ ; I say that the ratio of  $A$  to  $B$  will be greater than the ratio of  $C$  to  $D$ <sup>11</sup>."

will divide the given triangle into triangles whose areas are each either greater than or less than the area of half of the original triangle. This leads Leonardo to the consideration of the problem, to draw through a point, within a triangle and not on one of the medians, a line which will bisect the area of the triangle. Euclid, Prop. 19.

The last paragraph of Euclid's proof, as it has come down to us through Arabian sources, does not ring true, and it was not in the Euclidean manner to consider special cases.

After Leonardo's proof of Proposition 19, a numerical example is given.

<sup>10</sup> Leonardo gives the details of the proof for the case of one-third and does not refer to any other fraction. If, however, the "certain fraction" were the ratio of the lengths of two given lines,  $m:n$ , we could readily construct a rectangle equal to  $\frac{m}{n} \cdot AB \cdot BC$ , and then find the rectangle  $BZ \cdot ED$  equal to it. The rest of the construction is the same as given above.

According to the conditions set forth in Proposition 18, there will be two, one, or no solutions of Propositions 19 and 20. Leonardo considers only the Euclidean cases. Cf. notes 104 and 107.

The case where there is no solution may be readily indicated. Suppose, in the above figure, that  $BE = EH$ , then of all triangles formed by lines drawn through  $D$  to meet  $AB$  and  $BC$ , the triangle  $HBT$  has the minimum area. Easily shown synthetically as in D. CRESSWELL, *An Elementary Treatise on the Geometrical and Algebraical Investigations of Maxima and Minima*. Second edition, Cambridge, 1817, pp. 15-17. Similar minimum triangles may be found in connection with the pairs of sides  $AB, AC$  and  $AC, CB$ . Suppose that neither of these triangles is less than the triangle  $HBT$ . Then if

$$\Delta HBT : \Delta ABC > m : n,$$

the solution of the problem is impossible.

<sup>11</sup> This and the next four auxiliary propositions for which I supply possible proofs, seem to be neither formally stated nor proved by Leonardo. At least some of the results are nevertheless assumed in his discussion of Euclid's later propositions, as we shall presently see. Although the auxiliary propositions are not

Given  $A, D > B, C$ . To prove  $A + B < C + D$ .

Let the lines  $A, D$  be adjacent sides of a rectangle; and let there be another rectangle with side  $B$  lying along  $A$  and side  $C$  along  $D$ . Then either  $A$  is greater than  $B$ , or  $D$  greater than  $C$ , for otherwise the rectangle  $A, D$  would not be greater than the rectangle  $B, C$ .

given in the *Elements*, they are assumed as known by Archimedes, Ptolemy and Apollonius.

For example, in Archimedes' "On Sphere and Cylinder," II. 9. Heiberg, ed. I, 1910, p. 227; Heath, ed. 1897, p. 90, Woepke 21 is used. See also Eutocius' Commentary, Archimedis *Opera omnia* ed. Heiberg, III, 1881, p. 257, etc., and Heiberg, *Questiones Archimedeae*, Hauniae, 1879, p. 45 f. For a possible application by Archimedes in his *Measurement of a circle* of what is practically equivalent to Woepke 24, see Heath's *Archimedes*, etc., 1897, p. xc.

The equivalent of Woepke 24 is assumed in the proof of a proposition given by Ptolemy, 87-165 A.D., in his *Synaxis*, vol. I, Heiberg edition, Leipzig, 1898, pp. 13-14. This in turn is tacitly assumed by Aristarchus of Samos circa 310-230 B.C., in his work *On the Sizes and Distances of the Sun and Moon* see Heath's edition *Aristarchus of Samos the Ancient Copernicus*, Oxford, 1913, pp. 367, 369, 377, 381, 389, 391.

As to the use of the auxiliary propositions in the two works *Proportional Section* and *On Cutting off a Space*, of Apollonius, we must refer to Pappus' account. Pappi Alexandrini *Collectionis*, ed. Hultsch, vol. II, 1877, pp. 684 ff.; Woepke 21, 22 occur on pp. 696-697; Woepke 24 enters on pp. 684-687; Woepke 23, 25 are given on pp. 687, 689. Perhaps this last statement should be modified: for whereas Euclid's propositions affirm that if

$$a:b \geq c:d, \quad a-b; b \geq c-d; d,$$

Pappus shows that if

$$a:b \geq c:d, \quad a-b; c-d; c, \quad c-d;$$

but these propositions are immediately followed by others which state that if

$$a:b \geq c:d, \text{ then } a:a'; d:d',$$

Below is given a list of the various restorations of the above-named works of Apollonius, based on the account of Pappus. By reference to these restorations the way in which the auxiliary propositions are used or avoided may be observed. We have already Art. 21 noticed a connection of Apollonius' work *On Cutting off a Space* with our subject under discussion. Some of these titles will therefore supplement the list given in the Appendix.

11. *Snellii R. F. περί λόγων ἀποτομῆς καὶ περί χωρίων ἀποτομῆς*. *Apollonii conicorum sectio geometria*. Lugodini, ex officina Platiniama Raphelengii, MD, CVII pp. 23.

More or less extensive editions of Snellius's work is given in:

*De conicis sectionibus libri octo*, ed. J. Heiberg, in *Styllogis et Synopsis et Index*, etc., *Opera mathematica*, Studii et operi P. M. Maass, Paris, MDC, XLIV, p. 389.

*Conicorum libri octo*, P. Hergone, Paris, 1634, tome I, pp. 809-904; also Paris, 1644.

*Proportional Section*, ed. J. Heiberg, in *Styllogis et Synopsis et Index*, etc., *Opera mathematica*, Studii et operi P. M. Maass, Paris, 1877, pp. 684-687.

*On Cutting off a Space*, ed. J. Heiberg, in *Styllogis et Synopsis et Index*, etc., *Opera mathematica*, Studii et operi P. M. Maass, Paris, 1877, pp. 684-687.

*On the Sizes and Distances of the Sun and Moon*, ed. J. Heiberg, in *Styllogis et Synopsis et Index*, etc., *Opera mathematica*, Studii et operi P. M. Maass, Paris, 1877, pp. 367, 369, 377, 381, 389, 391.

*On the Sizes and Distances of the Sun and Moon*, ed. J. Heiberg, in *Styllogis et Synopsis et Index*, etc., *Opera mathematica*, Studii et operi P. M. Maass, Paris, 1877, pp. 367, 369, 377, 381, 389, 391.

Let then  $A > B$ . To  $D$  apply the rectangle  $B \cdot C$  and we get a rectangle

$$A' \cdot D = B \cdot C; \quad [I. 44-45]$$

then  $A' : B = C : D. \quad [VII. 19]$

But since  $A > B, \quad A > A',$

$$A : B > A' : B; \quad [V. 8]$$

$$\therefore A : B > C : D. \quad [V. 13]$$

Q. E. D.

Pappus remarks: Conversely if  $A : B > C : D, A \cdot D > B \cdot C$ . The proof follows at once.

For, find  $A'$  such that  $A' : B = C : D;$

then  $A : B > A' : B,$

and  $A > A'$ . But  $A' \cdot D = B \cdot C. \therefore A \cdot D > B \cdot C. \quad Q. E. D.$

## PROPOSITION 22.

**43.** "And when the product of  $A$  and  $D$  is less than the product of  $B$  and  $C$ , then the ratio of  $A$  to  $B$  is less than the ratio of  $C$  to  $D$ ."

*Die Bücher des Apollonius von Perga de sectione spatii wiederhergestellt von Dr W. A. Diesterweg...Elberfeld, 1827...pp. vi+154+5 pl.*

*Des Apollonius von Perga zwei Bücher vom Raumschnitt. Ein Versuch in der alten Geometrie von A. Richter. Halberstadt, 1828, pp. xvi+105+9 pl.*

*Die Bücher des Apollonius von Perga de sectione spatii, analytisch bearbeitet und mit einem Anhang von mehreren Aufgaben ähnlicher Art versehen von M. G. Grabow...Frankfurt a. M., 1834, pp. 80+3 pl.*

*Geometrische Analysis enthaltend des Apollonius von Perga sectio rationis, spatii und determinata, nebst einem Anhang zu der letzten, neu bearbeitet von Prof. Dr Georg Paucker, Leipzig, 1837, pp. xii+167+9 pl.*

M. Chasles discovered that by means of the theory of involution a single method of solution could be applied to the main problem of the three books of Apollonius above mentioned. This solution was first published in *The Mathematician*, vol. III, Nov. 1848, pp. 201-202. This is reproduced by A. Wiegand in his *Die schwierigeren geometrischen Aufgaben aus des Herrn Prof. C. A. Jacobi Anhang zu Van Swinden's Elementen der Geometrie. Mit Ergänzungen englischer Mathematiker...Halle, 1849, pp. 148-149, and it appears at greater length in Chasles' *Traité de Géométrie supérieure*, Paris, 1852, pp. 216-218; 2<sup>e</sup> éd. 1880, pp. 202-204. It was no doubt Chasles who inspired *Die Elemente der projectivischen Geometrie in synthetischer Behandlung. Vorlesungen von H. Hankel, Leipzig, 1875, "Vierter Abschnitt, Aufgaben des Apollonius," pp. 128-145; "sectio rationis," pp. 128-138; "sectio spatii," pp. 138-140.**

*The "Three Sections," the "Tangencies" and a "Loci Problem" of Apollonius...by M. Gardiner, Melbourne, 1860. Reprinted from the Transactions of the Royal Society of Victoria, 1860-1861, v, 19-91+10 pl.*

*Die sectio rationis, sectio spatii und sectio determinata des Apollonius nebst einigen verwandten geometrischen Aufgaben von Fr. von Lohmann. Progr. Königsberg in d. N. 1882, pp. 16+1 pl.*

"Ueber die fünf Aufgaben des Apollonius," von L. F. Offerdinger. *Jahreshefte des Vereins für Math. u. Naturwiss. in Ulm a. D.* 1888, 1, 21-38; "Verhältniss-henre," pp. 25-25; "Flächenschnitt," pp. 26-27.

From the above proof we evidently have

$$C : D = A : B,$$

that is,

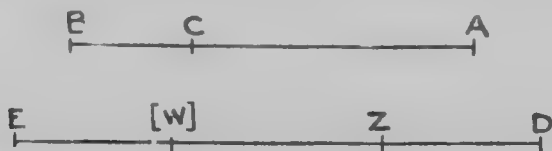
$$A : B = C : D$$

Conversely, as above, if  $A : B < C : D$ ,  $A : B < C : D$ .

It is really this converse, and not the proposition, which Euclid uses in Proposition 26. Proclus remarks (page 407) that the converses of Euclid's Elements, I. 35, 36, about parallelograms, are unnecessary "because it is easy to see that the method would be the same, and therefore the reader may properly be left to prove them for himself." No doubt similar comment is justifiable here.

### PROPOSITION 23.

44. "Given any two straight lines and on these lines the points  $A, B$ , and  $D, E$ ; and let the ratio of  $AB$  to  $BC$  be



greater than the ratio of  $DE : EZ$ ; I say that dividendo the ratio of  $AC$  to  $CB$  will be greater than the ratio of  $DZ$  to  $ZE$ ."

Given	$AB : BC > DE : EZ$	
To prove	$AC : CB > DZ : ZE$	
To $AB, BC, DE$ find a fourth proportional $EW$ .		[vi. 12]
Then	$AB : BC = DE : EW$	.....(1)
But	$AB : BC > DE : EZ$	
	$\therefore DE : EW < DE : EZ$	[v. 13]
	$\therefore EW < EZ$	[v. 8]
From (1)	$AC : CB = DW : WE$	.....(2) [v. 17]
since	$DW > DZ, DW : WE > DZ : WE$	[v. 8]
	$\therefore AC : CB > DZ : WE$	[v. 13]
But	$WE < ZE, \therefore DZ : WE > DZ : ZE$	[v. 8]
	$\therefore AC : CB > DZ : ZE$	From (2) at [v. 17]

Q. E. D.

## PROPOSITION 24.

45. "And in an exactly analogous manner I say that when the ratio of  $AC$  to  $CB$  is greater than the ratio of  $DZ$  to  $ZE$ , we shall have componendo<sup>112</sup> the ratio of  $AB$  to  $BC$  is greater than the ratio of  $DE$  to  $EZ$ ."

Given  $AC : CB > DZ : ZE$ .

To prove  $AB : BC > DE : EZ$ .

Determine  $W$ , as before, such that

$$AB : BC = DE : EW.$$

Then  $AC : CB = DW : WE$ . [v. 17]

$$\therefore DW : WE > DZ : ZE. \dots\dots(1) \quad [v. 13]$$

Now either  $EW > EZ$  or  $EW = EZ$ .

if  $EW > EZ$ ,  $DW < DZ$ , and

$$DW : EW < DZ : EW. \quad [v. 8]$$

So much the more is

$$DW : EW < DZ : EZ. \quad [v. 8]$$

which contradicts (1).

$$\therefore EW = EZ.$$

But  $AB : BC = DE : EW$ .

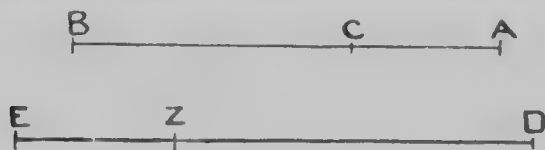
and  $DE : EW > DE : EZ$ ; [v. 8]

$$\therefore AB : BC > DE : EZ. \quad [v. 13]$$

Q. E. D.

## PROPOSITION 25.

46. "Suppose again that the ratio of  $AB$  to  $BC$  were



less than the ratio of  $DE$  to  $EZ$ ; dividendo the ratio of  $AC$  to  $CB$  will be less than the ratio of  $DE$  to  $ZE$ .<sup>113</sup>

<sup>112</sup> "Elements, Book V, definition 15" (Woepcke). This is definition 14 in HEATH, *The Thirteen Books of Euclid's Elements*, II, 135.

<sup>113</sup> The auxiliary propositions are introduced, apparently, to assist in rendering, with faultless logic, the remarkable proof of Proposition 26. In this proof it will be observed that we are referred back to Proposition 21, to the converse of Proposition 22 and to Proposition 23 only, although 23 is really the same as 25. But no step in the reasoning has led to Proposition 24. If this is unnecessary, why has it been introduced? [continued overleaf.]

Just as the proof of Proposition 22 was contained in that for Proposition 21, so here, the proof required is contained in the proof of Proposition 23. Similarly the converse of Proposition 25 flows out of 24.

## PROPOSITION 26.

47. "To divide a given triangle into two equal parts by a line drawn from a given point situated outside the triangle." [Leonardo 4, p. 116, ll. 35-36.]

Let the triangle be  $abg$  and  $d$  the point outside.

Join  $ad$  and let  $ad$  meet  $bg$  in  $e$ . If  $be = eg$ , what was required is done. For the triangles  $abe$ ,  $ceg$  being on equal bases and of the same altitude are equal in area.

But if  $be$  be not equal to  $eg$ , let it be greater, and draw through  $d$ , parallel to  $bg$ , a line meeting  $ab$  produced in  $z$ .

Since  $be > \frac{1}{2}bg$ ,

$$\text{area } ab \cdot be > \frac{1}{2} \text{area } ab \cdot bg; \quad [\text{Cf. VII. 17}]$$

much more then is

$$\text{area } ab \cdot zd > \frac{1}{2} \text{area } ab \cdot bg, \quad \text{since } zd > be.$$

Now take

$$\text{area } ib \cdot zd = \frac{1}{2} \text{area } ab \cdot bg; \quad [\text{I. 44}]$$

then

$$\text{area } ab \cdot be > \text{area } ib \cdot zd,$$

and

$$zd : be < ba : bi^{iii}, \quad [\text{Prop. 21 or 22}]$$

To answer this question, let us inspect the auxiliary propositions more closely. In a sense Propositions 21 and 22 go together: If  $ad \leq be$ , then  $a : b \geq c : d$ . So also for Propositions 23 and 25: If  $a : b \geq c : d$ , then  $a - b : b \geq c - d : d$ . Proposition 24 is really the converse of 23: If  $a : b > c : d$ , then  $a + b : b > c + d : d$ . Had Euclid given another proposition: If  $a : b < c : d$  then  $a + b : b < c + d : d$ , we should have had two groups of propositions 21, 22, and 23, 25 with their converses. Now the converses of 21 and 22 are exceedingly evident in both statement and proof. But this can hardly be said of the proof of 24, the converse of 23. The converse of 23 having been given the formulation of the statement and proof of the converse of 25 is obvious and unnecessary to state, according to Euclid's ideals (cf. Art. 43). It might therefore seem that Proposition 24 is merely given to complete what is not altogether obvious, in connection with the statement of the four propositions 21 and 22, 23 and 25, and their converses. In Pappus' discussion some support is given to this view, since Propositions 21 and 22 and converses are treated as a single proposition: Propositions 23, 25 as another proposition, while the converses of 23 and 25 are dealt with separately.

The more probable explanation is, however, that Propositions 23 and 24 were given by Euclid because they were necessary for the discussion of other cases of Proposition 26 assuming that the first case of Leonardo was that given by Euclid, for it was not his manner to consider different cases. Indeed if we take  $be$  less than  $ge$  in the first part of Leonardo's discussion exactly Propositions 23 and 24 are necessary.

<sup>iii</sup> Therefore  $bi < ba$ , and if  $bi$  be measured along  $ba$ ,  $i$  will fall between  $b$  and  $a$ .

But

$$zd : bc = za : ab,$$

[VI. 4]

$$\therefore zb : ba < ai : ib; \quad [\text{v. 13 and Prop. 25}]$$

or

$$\text{area } zb \cdot bi < \text{area } ba \cdot ai.$$

[Converse of Prop. 22]

Apply a rectangle equal to the rectangle  $zb \cdot bi$  to the line  $bi$ , but exceeding by a square<sup>115</sup>; that is to  $bi$  apply a line such that when multiplied by itself and by  $bi$  the sum will be equal to the product of  $zb$  and  $bi$ ; let  $ti$  be the side of the square<sup>116</sup>.

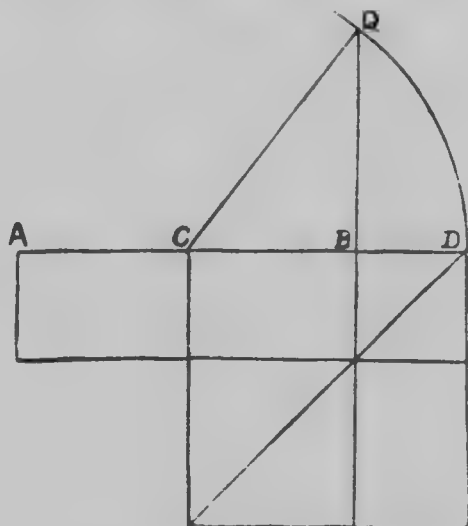
Draw the straight line  $tkd$ . Since

$$\text{area } zb \cdot bi = bi \cdot ti + ti^2 = \text{area } bt \cdot ti.$$

<sup>115</sup> Here again we have an expression with the true Greek ring: "adiungatur quidem recte  $bi$ , paralilogramum superhabundans figura tetragona equale superficie  $zb$ , in  $bi$ ."

<sup>116</sup> We have seen that  $i$  lies between  $b$  and  $a$ . And since it has been shown that  $zb \cdot bi < ba \cdot ai$ , we now have  $ba \cdot ai > bt \cdot ti$ . If  $bt > ba$ ,  $t$  is also greater than  $ai$ , and  $bt \cdot ti < ba \cdot ai$ . Therefore  $bt < ba$  and  $t$  falls between  $b$  and  $a$ . But it also falls between  $a$  and  $i$  by reason of the construction, always possible: which is called for.

In his book on *Divisions of figures*, Euclid does not formulate the proposition here quoted, possibly because of its similarity to Proposition 18 (see note 103).



If we let the rectangle  $zb \cdot bi = c^2$ ,  $ti = x$ , and  $bi = a$ , we have to solve geometrically the quadratic equation:

$$ax + x^2 = c^2.$$

[continued overleaf.]

$$b : bt = ti : ib, \quad [\text{VII. 19}]$$

or  $zt : bt = bt : bi, \quad [\text{V. 18}]$

But  $zt : bt = zd : bk, \quad [\text{VI. 4}]$

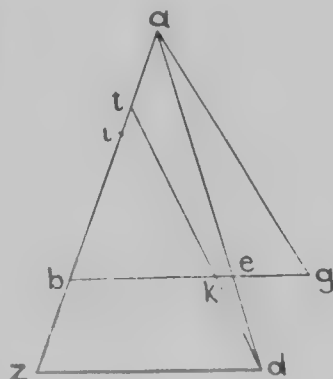
$$\therefore zd : bk = bt : bi,$$

and area  $kb \cdot bt = \text{area } zd \cdot bi$ .

But area  $zd \cdot bi = \frac{1}{2} \text{area } ab \cdot bg$ ,

$$\therefore \text{area } tbk = \frac{1}{2} \text{area } abg.$$

Therefore the triangle  $abg$  is divided by a line drawn from the point  $d$ , that is, by the line  $tkd$ , into two equal parts one of which is the triangle  $tbk$ , and the other the quadrilateral  $tkga$ .



Q. E. F.

Leonardo now gives a numerical example. He then continues :

Heath points out *Elements*, vol. I, pp. 386-387, that the solution of a problem theoretically equivalent to the solution of a quadratic equation of this kind is presupposed in the fragment of Hippocrates' *Quadrature of lines* (5th century B.C.) preserved in a quotation by Simplicius (fl. 500 A.D.) from Eudemus' *History of Geometry* (4th century B.C.). See Simplicius' *Comment. in Aristot. Phys.*, ed. H. Diels, Berlin, 1882, pp. 61-68; see also F. Rudio, *Der Bericht des Simplicius über die Quadratur des Antiphen und Hippokrates*, Leipzig, 1907.

Moreover as Proposition 18 is suggested by the *Elements*, II. 5, so here this problem is suggested by II. 6: *If a straight line be bisected and a straight line be added to it in a straight line, the rectangle contained by the whole with the added straight line and the added straight line together with the square on the half is equal to the square on the straight line made up of the half and the added straight line.*

If  $AB$  is the straight line bisected at  $C$  and  $BD$  is the straight line added, then by II. 6,

$$AD \cdot DB + CB^2 = CD^2.$$

In his solution of our problem, Robert Simson proceeds, in effect, as follows *Elements of Euclid*, ninth ed., Edinburgh, 1793, p. 336: Draw  $BQ$  at right angles to  $AB$  and equal to  $c$ . Join  $CQ$  and describe a circle with centre  $C$  and radius  $CQ$  cutting  $AB$  produced in  $D$ . Then  $BD$  or  $x$  is found. For, by II. 6,

$$AD \cdot DB + CB^2 = CD^2,$$

$$= CQ^2,$$

$$= CB^2 + BQ^2,$$

$$\therefore AD \cdot DB = BQ^2,$$

whence

$$(a+x) \cdot x = c^2$$

or

$$ax + a^2 = c^2.$$

It was not Euclid's manner to consider more than one solution in this case.

[If the point  $d$  were on one side,  $ab$ , produced at say,  $z$ ], through  $z$  draw  $ze$  parallel to  $bg$  and meeting  $ag$  produced in  $e$ .

Make

$$\text{area } ze . gi = \frac{1}{2} \text{area } ag . gb,$$

and apply a rectangle, equal to the rectangle  $eg . gi$ , to the line  $gi$ , but exceeded by a square;

then

$$eg . gi = gt . ti.$$

Join  $tz$ , then [this is the required line. The proof is step for step as in the first case].

Leonardo then remarks: "Que etiam demonstrentur in numeris," and proceeds to a numerical example. Thereafter he continues:

But let the sides  $ab$ ,  $gb$  of the triangle be produced to  $d$  and  $e$  respectively; and let  $i$  be the given point in the angle  $cbd$  from which a line is to be drawn dividing the triangle  $abg$  into two equal parts. Join  $ib$  and produce it to meet  $ag$  in  $z$ . If  $az = zg$ , the triangle  $abg$  is divided into two equal parts by the line  $iz$ . But [if  $az > zg$ ,] let  $za$  produced meet, in the point  $t$ , the line drawn through  $i$  parallel to  $ab$ .

Since

$$za > \frac{1}{2} ag, \quad \text{area } ab . az > \frac{1}{2} \text{area } ba . ag.$$

Make

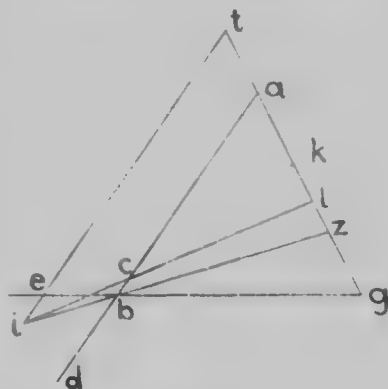
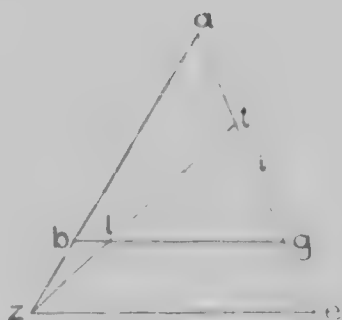
$$\text{area } it . ak = \frac{1}{2} \text{area } ba . ag;$$

then make

$$\text{area } al . kl = \text{area } ta . ak.$$

Join  $il$ . Then as above the triangle  $abg$  is divided into two equal parts by the line  $il$ , one part the triangle  $lac$ , the other the quadrilateral  $lcbg$ .

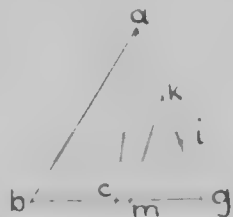
To this statement Leonardo adds nothing further. The proof that  $k$  lies between  $a$  and  $z$ , and  $l$  between  $k$  and  $z$ , follows as in the first part.



## PROPOSITION 27.

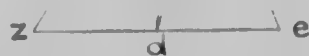
48. "To cut off a certain fraction of a triangle by a straight line drawn from a given point situated outside of the triangle<sup>48</sup>." [Leonardo II, p. 121, ll. 22-23.]

Let  $abg$  be the given triangle and  $d$  the given point outside. It is required to cut off from the triangle a certain fraction, say, one-third, by a line drawn through  $d$ . Join  $ad$ , cutting  $bg$  in  $c$ . If either  $bc$  or  $cg$  be one-third of  $bg$ , then the line  $ad$  through the point  $d$  cuts off one-third of the triangle  $abg$ . But if this be not the case produce  $ab$ ,  $ag$  to meet in  $z$  and  $e$  respectively the line drawn through  $d$  parallel to  $bg$ .



Make

$$\text{area } dc \cdot gi = \frac{1}{3} \text{ area } ag \cdot gb,$$



and apply to the line  $gi$  a rectangle

equal to the rectangle  $cg \cdot gi$ , but exceeded by a square; then

$$cg \cdot gi = ik \cdot kg.$$

Draw the line  $kmd$ . I say that the triangle  $kmg$  is one-third of the triangle  $abg$ .

*Proof:* For since

$$\text{area } cg \cdot gi = \text{area } gk \cdot ki,$$

$$cg : ki = ki : ig.$$

Hence

$$ck : gk = gk : gi.$$

[V. 17]

<sup>48</sup> Some generalizations of the triangle problems in Propositions 1, 2, 20, 26 and 27 may be remarked. Steiner, in 1827, solved the problem: *through a given point to pass a line in an arc of a great circle cutting two given great circles such that the intercepted area is equal to a given area.* (J. STEINER, "Verwandlung und Theilung sphärischen Figuren durch Construction," *Crelle Jh.* II, 1827, pp. 56 f. Cf. Syllabus of Townsend's course at Dublin Univ., 1846, in *Nouvelles Annales de Mathématiques*, Sept. 1850, IX, 364; also Question 427-7 proposed by Vannson in *Nouvelles Annales*, Jan. 1858, XVII, 45; answered Aug. 1859, XVIII, 335.) See also GUDERMANN, "Über die niedere Sphärik," *Crelle Jh.* 1832, VIII, 368.

In the next year Bobillier solved, by means of planes and spheres only, the problem, *to draw through a given line a plane which shall cut off from a given cone a solid of a given area.* (*Congress of Mathematicians and Physicists* [Quadrès], XI, 1855, p. 19, 25; Bobillier, 1827.)

But

$$ek : kg = de : gm ;$$

[VI. 2]

$$\therefore ed : gm = gk : gi.$$

$$\therefore \text{area } gk . gm = \text{area } de . gi.$$

But

$$\text{area } de . gi = \frac{1}{2} \text{area } ag . gb ;$$

$$\therefore \text{area } gk . gm = \frac{1}{2} \text{area } ag . gb.$$

And since

$$\text{area } gk . gm : \text{area } ag . gb = \angle kgm : \angle agb,$$

$$\angle kgm = \frac{1}{2} \angle agb.$$

In a similar manner any part of a triangle may be cut off by a straight line drawn from a given point, on a side of the triangle produced, or within two produced sides.

## PROPOSITION 28.

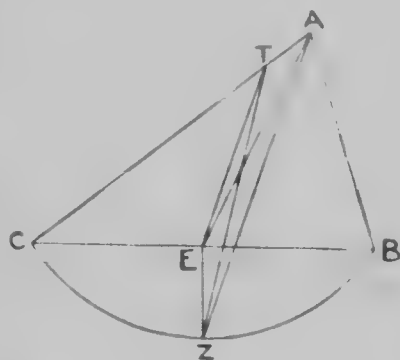
49 "To divide into two equal parts a given figure bounded by an arc of a circle and by two straight lines which form a given angle." [Leonardo 57, p. 148, ll. 13-14.]

"Let  $ABC$  be the given figure bounded by the arc  $BC$  and by the two lines  $AB, AC$  which form the angle  $BAC$ .

It is required to draw a straight line which will divide the figure  $ABC$  into two equal parts.

"Draw the line  $BC$  and bisect it at  $E$ . Through the point  $E$  draw a line perpendicular to  $BC$ , as  $EZ$ , and draw the line  $AE$ . Then because  $BE$  is equal to  $EC$ , the area  $BZE$  is equal to the area  $EZC$ , and the triangle  $ABE$

is equal to the triangle  $AEC$ . Then the figure  $ABZE$  will equal the figure  $ZCAE$ . If the line  $AE$  lie in  $EZ$  produced, the figure will be divided into two equal parts  $ABZE$  and  $CAEZ$ . But if the line  $AE$  be not in the line  $EZ$  produced, join  $A$  to  $Z$  by a straight line and through the point  $E$  draw a line, as  $ET$ , parallel to the line  $AZ$ . Finally draw the line





two equal parts at the point  $E$ , and draw from the point  $E$  the chord  $EZ$  parallel to the line  $BC$ . Finally draw the line  $AB$ . I say that we have two parallel lines  $EZ$ ,  $CB$  cutting off a third of the circle  $ABC$ , viz. the figure  $ZBC E$ .

"*Demonstration.* The line  $AC$  being parallel to the line  $DB$ , the triangle  $D, IC$  will be equal to the triangle  $B, IC$ . Add to each the segment of the circle  $IEC$ ; the whole figure  $D, IEC$  will be equal to the whole figure  $B, IEC$ . But the figure  $D, IEC$  is one-third of the circle. Consequently the figure  $B, IEC$  is also one-third of the circle. Since  $EZ$  is parallel to  $CB$ , the arc  $EC$  will be equal to the arc  $BZ$ ; but  $EC$  is equal to  $E, I$ , hence  $E, I$  equals  $ZB$ . Add to these equal parts the arc  $ECB$ ; the whole arc  $AB$  will equal the whole arc  $EZ$ . Consequently the line  $AB$  will be equal to the line  $EZ$ , and the segment of the circle  $IECB$  will be equal to the segment of the circle  $ECBZ$ . Taking away the common segment  $BC$ , there remains the figure  $E, ZBC$  equal to the figure  $B, IEC$ . But the figure  $B, IEC$  was one-third of the circle  $ABC$ . Then the figure  $E, ZBC$  is one-third of the circle  $ABC$ ; which was to be demonstrated.

"When it is required to cut off a quarter of a circle, or a fifth or any other definite fraction, by means of two parallel lines, we construct in this circle the side of a square or of the pentagon (regular) inscribed in the circle and we draw from the centre to the extremities of this side the two straight lines as above. (The remainder of) the construction will be analogous to that which has gone before".

The statement and form of discussion of this proposition are not wholly satisfactory. For "a certain fraction" in the enunciation we should rather expect "one-third," as in Leonardo; while at the conclusion of the proof might possibly occur a remark to the effect that a similar construction would apply when the certain fraction was one-quarter [by means of iv. 6], one-fifth [iv. 11], one-sixth [iv. 15], or one-fifteenth [iv. 16], but is it conceivable that Euclid added "or any other definite fraction"? Moreover the lack of definition of  $D$  and certain matters of form seem to further indicate that modification of the original has taken place in its passage through Arabian channels.

"This problem is clearly not susceptible of solution with ruler and compasses, in such a case as when the "certain fraction,"  $\frac{1}{n}$ , is one-seventh. In fact the only cases in which the problem is possible, for a fraction of this kind, is when  $n$  is of the form

$$2^p \cdot 2^{2^1} + 1 \cdot 2^{2^2} + 1 \dots 2^{2^m} + 1$$

where  $p$ , and  $s$ 's all different, are positive integers or zero, and  $2^{2^m} + 1$ ,  $m = 1, 2, \dots, n$  is a prime number. Cf. C. F. GAUSS, *Disquisitiones Arithmeticae*, Lipsiae, 1801 French ed., Paris, 1847, p. 127.

On the other hand Leonardo presents the proposition as if drawn from the pure well of Euclid undefiled. Here is his discussion. (I have substituted  $C$  for his  $b$ , and  $B$  for his  $g$ .)

"And if, by means of two parallel lines, we wish to cut off from a circle  $ACB$ , whose centre is  $D$ , a given part which is one-third, draw the line  $AC$ , the side of an equilateral triangle inscribed in the circle  $abg$ . Through the centre  $D$  draw  $DB$  parallel to this line and join  $CB$ . Bisect the arc  $AC$  at  $E$  and draw  $EZ$  parallel to  $bg$ . I say that the figure contained between the lines  $CB$  and  $EZ$  and the arcs  $EC$  and  $BZ$  is one-third part of the circle  $ACB$ .

"*Proof.* Draw the lines  $DA$  and  $DB$  and  $AB$ .

"The triangles  $BAC$  and  $DAC$  are equal. To each add the portion  $ABE$ . Then the figure bounded by the lines  $BA$  and  $BC$  and the arc  $AEC$  is equal to the sector  $DAEC$  which is a third part of the circle  $ABC$ .

"Therefore the figure bounded by the lines  $BA$  and  $BC$  and the arc  $AEC$  is a third part of the circle.

"And since the lines  $CB$  and  $EZ$  are parallel, the arcs  $EC$  and  $BZ$  are equal. But arc  $EC$  is equal to arc  $AE$ . Therefore arc  $AE$  is equal to the arc  $BZ$ . To each add the arc  $EB$ , and then the arc  $AECB$  will be equal to the arc  $ECBZ$ .

"Hence the portion  $EZBC$  of the circle is equal to the portion  $ABCE$ . Take away the common part between the line  $CB$  and the arc  $BC$  and there remains the figure, bounded by the lines  $BC$  and  $EZ$  and the arcs  $CE$  and  $BZ$ , which is the third part of the circle since it is equal to the figure bounded by the lines  $BA$  and  $BC$  and the arc  $AEC$ ; quod oportebat ostendere.

In his *metron* (III, 18) Heron of Alexandria considers the problem: *To divide the area of a circle into three equal parts by two straight lines.* He remarks that "it is clear that the problem is not rational"; nevertheless "on account of its practical use" he proceeds to give an approximate solution. By discussion similar to that above he finds the figure  $BCEA$ , formed by the triangle  $BCA$  and the segment  $CEA$ , to be one third of the circle. Neglecting the smaller segment with chord  $BC$ , we have, that  $BA$  cuts off "approximately" one-third of the circle. Similarly a second chord from  $B$  might be drawn to cut off another third of the circle, and the approximate solution be completed.

## PROPOSITION 30.

51. "To divide a given triangle into two parts by a line parallel to its base, such that the ratio of one of the two parts to the other is equal to a given ratio."

Although Leonardo does not explicitly formulate this problem or the next, the method to be employed is clearly indicated in the discussion of Proposition 5 (Art. 26).

Let  $abg$  be the triangle which is to be divided in the given ratio  $ez : zi$ , by a line parallel to  $bg$ . Divide  $ab$  in  $h$  such that

$$ah^2 : ab^2 = ez : ei^2.$$

Draw  $hk$  parallel to  $bg$  and meeting  $ag$  in  $k$ . Then the triangles  $ahk$  and  $abg$  are similar and

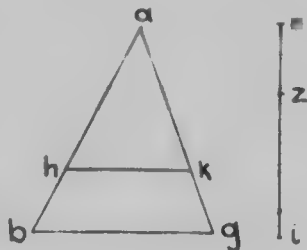
$$\triangle ahk : \triangle abg = ah^2 : ab^2. \quad [\text{VI. 19}]$$

But  $ah^2 : ab^2 = ez : ei$ ,

$$\therefore \triangle ahk : \triangle abg = ez : ei;$$

whence  $\triangle ahk : \text{quadr. } hb gk = ez : zi; \quad [\text{V. 16, 17}]$

and the triangle  $abg$  has been divided as required.



## PROPOSITION 31.

52. "To divide a given triangle by lines parallel to its base into parts which have given ratios to one another."

Again in the manner of Proposition 5, suppose it be required to divide the triangle  $abg$  into three parts in the ratio  $ez : st : ti$ . Then determine the points  $h, l$  in  $ab$  such that

$$ah^2 : ab^2 = ez : ei^2;$$

and  $al^2 : ab^2 = et : ei$ .

Then  $ah^2 : al^2 = ez : et. \quad [\text{V. 16, 20}]$

$$\therefore \triangle ahk : \triangle alm = ez : et.$$

and  $\therefore \triangle ahk : \text{quadr. } hlmk = ez : st.$

Similarly

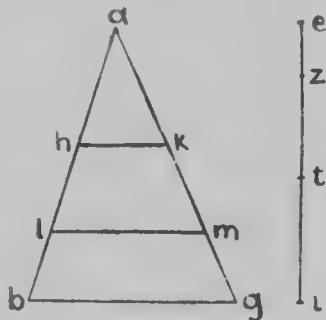
$$\triangle alm : \text{quadr. } lbgm = et : ti.$$

But  $\triangle ahk : \triangle alm = ez : et.$

$$\therefore \triangle ahk : \text{quadr. } lbgm = ez : ti. \quad [\text{V. 20}]$$

Hence,  $\triangle ahk : \text{quadr. } hlmk : \text{quadr. } lbgm = ez : st : ti,$

and the triangle  $abg$  has been divided into three parts in a given ratio to one another. So also for any number of parts which have given ratios to one another.



## PROPOSITION 32.

53. "To divide a given trapezium by a line parallel to its base, into two parts such that the ratio of one of these parts to the other is equal to a given ratio." [Leonardo 29, p. 131, ll. 41-42.]

Let  $abgd$  be the trapezium which is to be divided in the ratio  $ez : zi$  by a line parallel to the base. Produce the sides  $ba$ ,  $gd$  through  $a$  and  $d$  to meet in  $t$ .

Make  $tl^2 : at^2 = zi : ez$ ,  
and  $ht^2 : (bt^2 + tl^2) = ez : ei$ .

Through  $l$ ,  $h$ , draw  $lm$ ,  $hk$  parallel to  $bg$  and  $ad$ . Then I say that the quadrilateral  $ag$  is divided in the given ratio,  $ez : zi$ , by the line  $hk$ .

*Proof:* For since the triangles  $tlm$ ,  $tad$  are similar

$$tl^2 : at^2 = \triangle tlm : \triangle tad;$$

$$\text{but } tl^2 : at^2 = zi : ez;$$

$$\therefore zi : ez = \triangle tlm : \triangle tad.$$

$$\text{Whence } ei : ez = (\triangle tlm + \triangle tad) : \triangle tad, \quad [\text{v. 18}]$$

$$\text{or } ez : ei = \triangle tad : (\triangle tlm + \triangle tad). \quad [\text{v. 16}]$$

But by construction

$$ez : ei = ht^2 : (bt^2 + tl^2),$$

$$\text{and } ht^2 : (bt^2 + tl^2) = \triangle thk : (\triangle tbg + \triangle tlm). \quad [\text{vi. 19}]$$

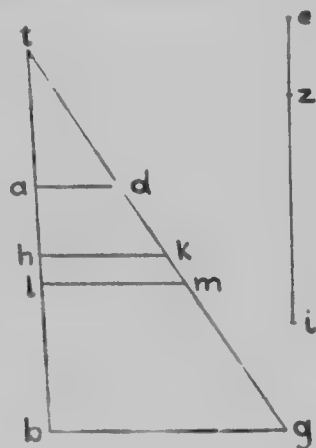
$$\therefore ez : ei = \triangle thk : (\triangle tbg + \triangle tlm).$$

$$\text{But } \triangle thk = \triangle tad + \text{quadr. } ak.$$

Similarly

$$\triangle tbg + \triangle tlm = \text{quadr. } ag + \triangle tad + \triangle tlm.$$

$$\therefore ez : ei = (\text{quadr. } ak + \triangle tad) : (\text{quadr. } ag + \triangle tad + \triangle tlm).$$



But  $ez : ei = \triangle tad : (\triangle tad + \triangle tlm)$ ;

$$\therefore ez : ei = \text{quadr. } ak : \text{quadr. } ag; \quad [\text{v. 11, 19}]$$

whence  $ez : zi = \text{quadr. } ak : \text{quadr. } hg$ .

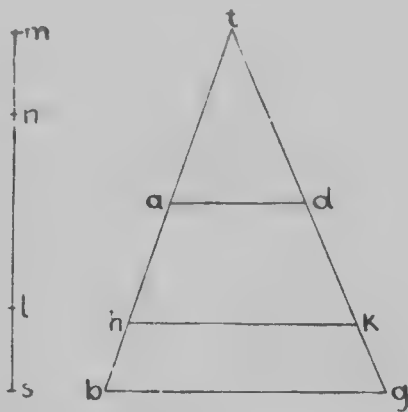
And the trapezium has been divided in the given ratio.

Then follows a numerical example and this alternative construction and proof:

Draw  $mls$  such that,

$$ms : ls = tb^2 : ta^2,$$

and divide  $ml$  in  $n$ , such that  $ln$  is to  $nm$  in the given proportion.



In  $tb$  determine  $h$  such that

$$th^2 : tb^2 = ns : sm.$$

Draw  $hk$   $bg$ . Then,

$$\text{quadr. } ak : \text{quadr. } hg = ln : nm.$$

*Proof:* For,  $tb^2 : ta^2 = \triangle tbg : \triangle tad$ ; [vi. 19]

and  $ms : ls = tb^2 : ta^2$ .

$$\therefore ms : ls = \triangle tbg : \triangle tad. \quad \dots\dots\dots[1]$$

Again, since  $tb^2 : th^2 = ms : sn$ ,

and  $\triangle tbg : \triangle thk = tb^2 : th^2$ ,

$$\therefore ms : ns = \triangle tbg : \triangle thk; \quad \dots\dots\dots[2]$$

$$\therefore sm : nm = \triangle tbg : \text{quadr. } hg. \quad [\text{v. 16, 21}]$$

\dots\dots\dots[3]

But  $sm : ls = \triangle tbgr : \triangle tad$ ,  
 or  $ms : \triangle tbgr = ls : \triangle tad$ ,  
 while  $ms : \triangle tbgr = ns : \triangle thk$ ; [from [2]]  
 $\therefore ls : ns = \triangle tad : \triangle thk$ . ..... [4]  
 From [3]  $ms : \triangle tbgr = nm : \text{quadl. } hg$ .  
 But from [4]  $sl : ln = \triangle tad : \text{quadl. } ak$ , [v. 16, 21]  
 $\therefore sl : \triangle tad = ln : \text{quadl. } ak$ .  
 But from [1]  $ms : \triangle tbgr = sl : \triangle tad$ ,  
 $\therefore ms : \triangle tbgr = ln : \text{quadl. } ak$ ;  
 $\therefore mn : \text{quadl. } hg = ln : \text{quadl. } ak$ ;  
 $\therefore ln : nm = \text{quadl. } ak : \text{quadl. } hg$ .

Hence the quadrilateral  $ag$  is divided by the line  $hk$ , parallel to the base  $bg$ , in the given proportion as the number  $ln$  is to the number  $nm$ . Which was to be done.

Then follows a numerical example.

### PROPOSITION 33.

54. "To divide a given trapezium, by lines parallel to its base, into parts which have given ratios to one another."  
 [Leonardo 35, p. 137, ll. 6-7.]

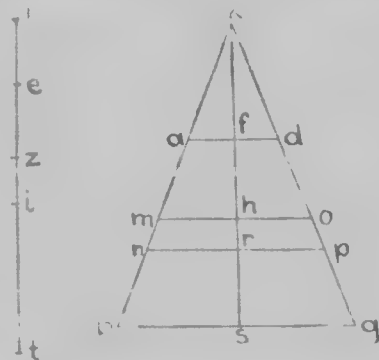
Let  $abcd$  be the given trapezium and [let  $ez : zi : il$  denote the ratios of the three parts into which the trapezium is to be divided by lines parallel to the base  $bg$ ]. Let  $ba, gd$  produced meet in  $k$  and find  $l$  such that

$$bk^2 : ak^2 = tl : cl.$$

Then determine  $m$  and  $n$  such that

$$bk^2 : km^2 = tl : lz,$$

and  $bk^2 : kn^2 = tl : il$ .



<sup>111</sup> Such mixed ratios as these (ratios of lines to areas), and others of like kind which follow in this proof, are very un-Greek in their formation. This is sufficient to stamp the second proof as of origin other than Greek. The first proof, on the other hand, is purely Euclidean in character.

Through  $m, n$  draw lines  $mo, np$  parallel to  $bg$ . In the same manner as above

$$\text{quadr. } ao : \text{quadr. } mp = ez : zi;$$

and

$$\text{quadr. } mp : \text{quadr. } ng = zi : it.$$

There follows a numerical example in which the line  $kphs$ , perpendicular to  $bg$ , is introduced into the figure.

Here is a proof of the Proposition:

By construction, v. 16 and vi. 10,

$$\triangle kbg : \triangle kad = tl : el. \dots\dots\dots [1]$$

So also

$$\triangle kbg : \triangle kmo = tl : lz. \dots\dots\dots [2]$$

and

$$\triangle kbg : \triangle knp = tl : il. \dots\dots\dots [3]$$

From [1]

$$\triangle kad : \triangle kbg = el : tl.$$

But from [2]

$$\triangle kbg : \triangle kmo = tl : lz;$$

hence, by [v. 20],

$$\triangle kad : \triangle kmo = el : lz. \dots\dots\dots [4]$$

or alternately

$$\triangle kmo : \triangle kad = lz : el.$$

Hence, *separando*,

$$\text{quadr. } ao : \triangle kad = ez : el. \dots\dots\dots [5]$$

So also from [2] and [3]

$$\triangle kmo : \triangle knp = lz : il;$$

and

$$\triangle kmo : \text{quadr. } mp = lz : iz.$$

But from [4]

$$\triangle kad : \triangle kmo = el : lz;$$

therefore, by [v. 20],

$$\triangle kad : \text{quadr. } mp = el : iz. \dots\dots\dots [6]$$

Hence from [5], by [v. 20],

$$\text{quadr. } ao : \text{quadr. } mp = ez : zi.$$

Again, from [3],

$$\text{quadr. } ng : \triangle kbg = ti : tl;$$

and since from [1],

$$\triangle kbg : \triangle kad = tl : el,$$

we have

$$\text{quadr. } ng : \triangle kad = ti : el.$$

Hence from [6], by [v. 20], we get

$$\text{quadr. } ng : \text{quadr. } mp = it : zi,$$

or alternately

$$\text{quadr. } mp : \text{quadr. } ng = zi : it$$

And since  $\text{quadr. } ao : \text{quadr. } mp = ez : zi$ , the trapezium  $ag$  has been divided by lines parallel to the base  $ag$ , into three parts which are in the required ratios to one another.

Q. E. F.

## PROPOSITION 34.

55. "To divide a given quadrilateral, by a line drawn from a given vertex of the quadrilateral, into two parts such that the ratio of one of these parts to the other is equal to a given ratio." [Leonardo 40, p. 140, ll. 36-37.]

Let  $abcd$  be the given quadrilateral, and  $ez : zi$  the given ratio. It is required to draw from the angle  $d$  a line to divide the quadrilateral in the ratio  $ez : zi$ .

Draw the diagonal  $ac$  and on it find  $t$  such that

$$ct : at = ez : zi.$$

Draw the diagonal  $bd$ . Then if  $bd$  pass through  $t$  the quadrilateral is divided as required, in the ratio  $ez : zi$ .

For,

$$\begin{aligned} \triangle dct : \triangle dta &= ct : ta, \\ &= \triangle cdt : \triangle tba, \\ \therefore ct : ta &= \triangle dcb : \triangle abd. \quad [\text{v. 18}] \end{aligned}$$

But

$$\begin{aligned} ct : ta &= ez : zi; \\ \therefore ez : zi &= \triangle dcb : \triangle abd; \end{aligned}$$

and the quadrilateral  $ac$  is divided, by a line drawn from a given angle, in a given ratio.

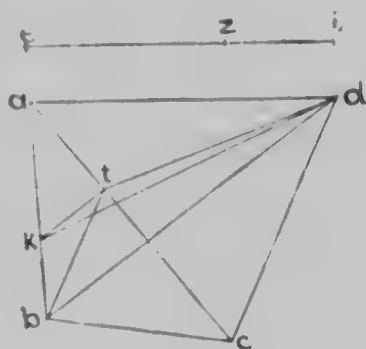
But if  $bd$  do not pass through  $t$ , it will cut  $ca$  either between  $c$  and  $t$  or between  $t$  and  $a$ . Consider first when  $bd$  cuts  $ct$ . Join  $bt$  and  $td$ . Then,

$$\text{quadr. } tbcd : \text{quadr. } tbad = ct : ta = ez : zi.$$

Draw  $tk \parallel bd$  and join  $dk$ . Then

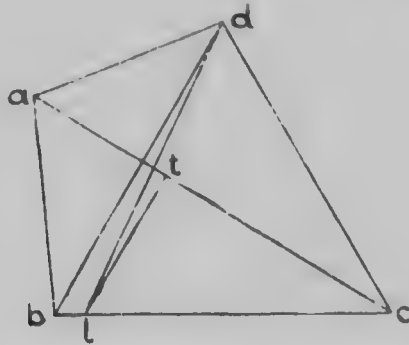
$$\begin{aligned} \text{quadr. } kbcd &= \text{quadr. } tbcd; \\ \therefore \text{quadr. } kbcd : \triangle dak &= ez : zi, \end{aligned}$$

and the line  $dk$  has been drawn as required.



If the diagonal  $bd$  cut  $at$ , through  $t$  draw  $ll$  parallel to the diagonal  $bd$ . Join  $dl$ . Then as before,

$$ct : ta = cz : zi = \triangle del : \text{quadl. } abld.$$



Hence in every case the quadrilateral has been divided as required by a line drawn from  $d$ . Similarly for any other vertex of the quadrilateral.

### PROPOSITION 35.

**56.** "To divide a given quadrilateral by lines drawn from a given vertex of the quadrilateral into parts which are in given ratios to one another."

Although Leonardo does not explicitly formulate this problem, the method he would have followed is clear from his discussion of the last Proposition. Let  $abcd$  be the quadrilateral to be divided, by lines drawn from  $d$ , into three parts in the ratios to one another of  $cz : zi : it$ .



Divide  $ca$  at points  $r, t$  so that

$$cr : rt : ta = cz : zi : it.$$

Through  $r, t$  draw lines parallel to  $bd$ , and meeting  $bc$  (or  $ab$ ) in  $l$  and  $ab$  (or  $bc$ ) in  $m$ .

Then as above  $dl, dm$  divide the quadrilateral as required.

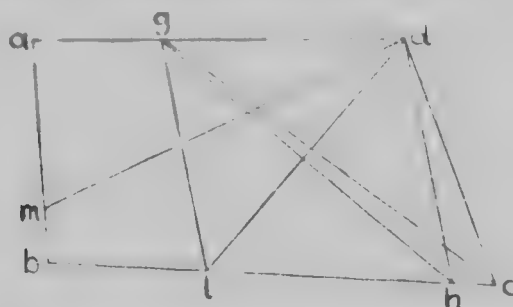
We may proceed in a similar manner to divide the quadrilateral  $abcd$ , by lines drawn from the angular point  $d$ , into any number of parts in given ratios to one another.

## PROPOSITION 36.

57. "Having resolved those problems which have gone before, we are in a position to divide a given quadrilateral in a given ratio or in given ratios by a line or by lines drawn from a given point situated on one of the sides of the quadrilateral, due regard being paid to the conditions mentioned above."

This problem, also, is not formulated by Leonardo; but from his discussion of Euclid's Propositions 16, 17 and of his own 41, the method of construction which Euclid might have employed are clearly somewhat as follows.

Let  $abcd$  be the given quadrilateral and  $g$  the given point.



(1) Let it be required to divide  $abcd$  into two parts in the ratio  $ez : zi$  by a line drawn through a point  $g$  in the side  $ad$ .

Draw  $dl$  such that  $\angle dld : \text{quadl. } lbad = ez : zi$ . [Prop. 34]

Join  $gl$ . If  $gl \parallel dc$ , join  $gc$ , then this line divides the quadrilateral as required.

If  $gl$  be not parallel to  $dc$  draw  $dh \parallel gl$ , and meeting  $bc$  in  $h$ . Join  $gh$ . Then  $gh$  divides the quadrilateral as required.

If  $dh$  fall outside the quadrilateral draw  $h'c \parallel ad$  (not indicated in the figure) to meet  $ad$  in  $l'$ . Draw  $l'g' \parallel gc$  to meet  $dc$  in  $g'$ . Then  $g'z'$  is the line required.

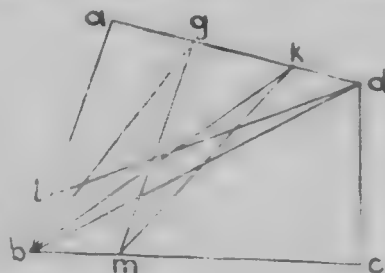
The above reasoning is on the assumption that  $dl$  meets  $bc$  in  $l$ . Suppose now it meet  $ab$  in  $l$ . Join  $bd$  and draw  $bk$  such that

$$\text{quadl. } l'abk : \text{quadl. } l'bad = ez : zi.$$

If  $k$  do not coincide with  $g$  there are two cases to consider according as  $k$  is between  $g$  and  $d$  or between  $g$  and  $a$ . Consider the former case.

Through  $k$  draw  $km$  parallel to  $l'$  and meeting  $bc$  in  $m$ . Join  $gm$ . Then this is the required line dividing the quadrilateral  $ac$  in such a way that

$$\text{quadl. } abmg : \text{quadl. } gmad$$



Similarly if  $k$  were between  $g$  and  $a$ .

(2) Let it be required to divide  $abcd$  into, say, three parts in the ratios  $e z : z i : i t$ , by lines through any point  $g$  in the side  $ad$  (first figure).

Draw  $dl, dm$  dividing the quadrilateral  $ac$  into three parts such that

$$mdl : \text{quadr. } dmbl : \triangle dlc = ez : zi : it.$$

There are various cases to consider according as  $l$  and  $m$  are both on  $bc$ , both on  $ab$ , or one on  $ab$  and one on  $bc$ . The method will be obvious from working out one case, say the last.

Join  $gc, gl$ . If  $gl$  be parallel to  $cd$ ,  $gc$  cuts off the triangle  $edc$  such that

$$gdc : \text{quadr. } abcg = it : et (= ez + zi). \quad [\text{v. 24}]$$

If  $gl$  be not parallel to  $cd$ , draw  $dh$  parallel to  $gl$  and meeting  $bc$  in  $h$ , then  $gh$  divides the quadrilateral in such a way that

$$\text{quadr. } gdc h : \text{quadr. } ghba = it : et.$$

Then apply Proposition 34 to draw from  $g$  a line to divide the quadrilateral  $abhg$  in the ratio of  $et : it$ .

Hence from  $g$  are drawn two lines which divide the quadrilateral  $abcd$  into three parts whose areas are in the ratios of  $ez : zi : it$ .

The case when  $dh$  meets  $bc$  produced may be considered as above.

We could proceed in a similar manner if the quadrilateral  $abcd$  were to be divided by lines drawn from  $g$ , into a greater number of parts in given ratios.

The enunciation of this proposition is a manifest corruption of what Euclid may have given. Such clauses as those at the beginning and end he would only have included in the discussion of the construction and proof.

After the enunciation of Proposition 36, Woepcke's translation of the Arabian MS. concludes as follows:

"End of the treatise. We have confined ourselves to giving the enunciations without the demonstrations, because the demonstrations are easy."

## IV.

## APPENDIX

In the same page, I have referred to works on Divisors of  $\mathbb{P}^2$  written before 1900. Several of these were republished later, for example that of Meunier (1900) in 1927, of Luroth (1869) in 1906, and the second edition of Luroth (1869) in 1923. It has been noted that Luroth's paper of 1869 was based upon a paper of Bézout's, but that the latter was published in 1861, and is already among the earliest sources of the program on

[illegible][illegible]

Propositions 19, 20, 2004-27, and 28 in the place referred to.

Thus,  $\mathcal{C}_1$  is a complement to  $\mathcal{C}_2$  in  $\mathcal{C}$  if and only if  $\mathcal{C}_1 \cap \mathcal{C}_2 = \emptyset$  and  $\mathcal{C}_1 \cup \mathcal{C}_2 = \mathcal{C}$ .

- [illegible]

1547. N. ESCOBAR. *Principios de la Geometría de A. Tarski*. *Revista de Matemática* 1 (1947), 1-17.

Dr. ESCOBAR, 1547, has written a book on the foundations of geometry, in which he discusses the work of A. Tarski.

He also discusses the work of H. Frege, and the work of L. Brouwer, and the work of J. van Fraassen.

He also discusses the work of K. Gödel, and the work of P. Cohen, and the work of S. Cohen.

He also discusses the work of G. Cantor, and the work of C. von Neumann, and the work of E. Schrödinger.

He also discusses the work of A. Einstein, and the work of N. Bohr, and the work of W. Heisenberg.

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1548. N. ESCOBAR. *Principios de la Geometría de A. Tarski*. *Revista de Matemática* 1 (1947), 1-17.

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He also discusses the work of R. Feynman, and the work of J. Schwinger, and the work of S. Weinberg.

1585. G. B. FENDELL. *Principios de la Geometría de A. Tarski*. *Revista de Matemática* 1 (1947), 1-17.

Dr. FENDELL, 1585, has written a book on the foundations of geometry, in which he discusses the work of A. Tarski.

He also discusses the work of H. Frege, and the work of L. Brouwer, and the work of J. van Fraassen.

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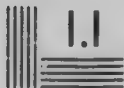
He also discusses the work of A. Einstein, and the work of N. Bohr, and the work of W. Heisenberg.

He also discusses the work of R. Feynman, and the work of J. Schwinger, and the work of S. Weinberg.

For CAN, see the book by G. B. Fendell, *Principios de la Geometría de A. Tarski*, 1947, pp. 1-17, 450-451, where the work of A. Tarski is discussed.



MICROCOPY RESOLUTION TEST CHART



28

25

22

22



2.0



1.8



4.0

3.6



- 1674—C. F. M. DESCHALES. *Cursus seu mundus mathematicus*. Lugduni.  
"De quadratum planum divisione," I, 371-381; second edition, 1690.
- 1676—I. NEWTON. *Arithmetica Universalis*. Cantabrigiae, MDCCVII.  
Prob. V, p. 126 (Prob. XX, pp. 254-255 of the 1769 edition). This problem was discussed in a lecture delivered October, 1676 (see *Curve Pontence of Sir Isaac Newton and His Followers*, by J. Edleston, London, 1850, p. xxviii).
- 1684—T. STRODE. *A Discourse of Combinations, Alternations and Aliquod Paris* by John Wallis. London, 1685.  
On pp. 163-164 is printed a letter, dated Nov. 1684, from Strode to Wallis. It discusses two problems on divisions of a triangle.
- 1687—J. BERNOUILLI. "Solutio algebraica problematis de quadrisectione trianguli scaleni, per duas normales rectas." *Acta Eruditorum*, 1687, pp. 617-623.  
Also in *C. 707*, GENÈVE, 1744, I, 328, 335; see further II, 671. In the solution of this question Bernoulli is led to the intersection of a circle and a curve of the fourth degree, that is, to an equation of the eighth degree. And yet, in the seventh edition of ROOTH and C. MEYER-SSER, *Traktat über die Algebra*, Part. 1600 we find Problem 453 is: "Partager un triangle quelconque en quatre parties équivalentes par deux droites perpendiculaires entre elles." The problem was solved by l'Hôpital before 1704, the year of his death, in a posthumous work, *Leçons sur l'usage du compas*, Paris, 1707, pp. 400-407. As the result of correspondence in *Le Journal de Trévoux*, I, 1711, 1894-1900, Questions 3 and 587, I have written the history of the problem. "1687 Bernoulli's problem is not a problem per se, but a problem per se." *Rev. Math.* 1903, I, IV, 48-51. Bernoulli's name appears in this connection. See note on 1645—C. H. 3218.
- 1688—J. OZANAM. *L'usage du compas de proportion expliqué et démontré d'une manière court et facile, et augmenté d'un Traité de la division des champs*. Paris.  
"Division des champs," pp. 89-138. The title is in orange and entirely reton lu par J. G. Garnier. Paris, 1794, pp. 165-257.
- 1694—S. LE CLERC. *Traité de Géométrie sur le terrain at end of Géométrie pratique, ou pratique de la géométrie sur le papier et sur le terrain*. Amsterdam.
- 1699—J. OZANAM. *Cours de mathématique*, nouv. éd. tome 3. Paris.  
Pages 23-64. German translation: *Anweisung, die geradlinichten Figuren nach ihren Theilen von Verhältnissen ohne Rechnung zu theilen*. Frankfurt u. Leipzig, 1776.
- 1704—GUINÉE. *Application de l'algèbre à la géométrie*. Paris.  
Although the "approbation" signed by Fontenelle is dated "15 Juillet 1704" the work was first published in 1710; second edition "révisé, corrigé et considérablement augmentée par l'auteur," Paris, 1733, pp. 42-47; analytic discussion only.
- 1739—L'abbé DEIDIER. *La science des géomètres (sic) ou la théorie et la pratique de la géométrie*. Paris.  
"De la géodésie ou division des champs," pp. 279-320; divisions of triangles, rectangles, trapeziums, polygons.
- 1740—N. SAUNDERSON. *Elements of Algebra in ten books*, vol. 2. Cambridge.  
Pages 546-554.
- 1747—T. SIMPSON. *Elements of Plane Geometry*. London.  
Pages 151-152; new ed., London, 1821, pp. 207-208; taken from Newton (1676).



- ... lines be drawn from it to cut each side at right angles, the parts into which the triangle thus becomes divided, shall obtain a given ratio." Solution by the periphrasis in No. 55, 1795, pp. 37-38. See also Davis's edition of the *Gentleman's Diary*, vol. 3, London, 1814, pp. 117-118.
- 1801—L. PUISSANT. *Recueil de divers propositions de géométrie résolues ou démontrées par l'analyse algébrique suivant les principes de Monge et de Legendre*. Paris.
- 1801—L. PUISSANT. *Recueil de divers propositions de géométrie résolues ou démontrées par l'analyse algébrique suivant les principes de Monge et de Legendre*. Berlin, 1806; second French ed., Paris, 1809, pp. 107-111; 3rd ed., Paris, 1824, pp. 131-142.
- 1805—M. HIRSCH. *Sammlung geometrischer Aufgaben, Erster Theil*. Berlin.
- "Theilung der Figuren nach Zeichnung," pp. 14-25; "Theilung der Figuren nach Inhalt," pp. 42-53; Leipzig, 1827. English translation of vol. I, J. A. Kress, ed., 1861; vol. II, M. F. Wright, London, 1827.
- 1807—A. BRAFF. *Problema geometricum triangulum datum a dato puncto in 2 partes aequales secandi*. Greifswald.
- This title is taken from C. G. KAYSER, *Bücher-Lexicon*, Erster Teil, Leipzig, 1834.
- 1807—J. P. CARLMARK. *Triangulus datus a dato puncto in 2 partes aequales secandus*. Greifswald.
- This title and the next two are taken from E. WOLFFING, *Math. Bucherschatz*, 1903.
- 1809—J. KULLBERG. *Problema geometricum triangulum datum e quovis dato puncto in 2 partes aequales secandi*. Diss. Lund.
- 1810—J. KULLBERG. *Problema geometricum triangulum quodcumque datum in 2 aequales divisum iterum in partes aequales ita secandi, ut rectae secantes angulum constituent rectum*. Diss. Upsala.
- 1811—L. P. GRISON. *Geodasie oder vollständige Anleitung zur geometrischen und ökonomischen Feldertheilung*. Halle.
- 1819—L. BLEIBEREU. *Theilungslehre oder ausführliche Anleitung, die Grundfläche auf die zweckmassigste Art .. geometrisch zu theilen*. Frankfurt am Main.
- 1821—J. LESLIE. *Geometrical Analysis and Geometry of Curve Lines*. Edinburgh.
- Pages 64-66.
- 1823—A. K. P. VON FORSTNER. *Sammlung systematisch geordneter und synthetisch aufgelöseter geometrischer Aufgaben*. Berlin.
- "Theilung der Flächen, mittelst der Proportion und der Aehnlichkeit," pp. 310-371.
- 1827—*Correspondance mathématique et physique* publié par A. Quetelet, tome III.
- Page 180: "On donne dans un plan un angle et un point, et l'on demande de faire passer par le point une droite qui coupe les côtés de l'angle, de manière que l'aire intérieure soit de 21 (ou de 22) de la donnée." Solution by Verhulst, pp. 269-270. Answer by Boscovich, tome IV, pp. 2-3. Generalizing his solution, he gets the result: "Tous les plans tangents d'un hyperboloïde à deux nappes, interceptent sur le cône asymptotique des sections équivalentes." Compare note 117.
- 1831—P. L. M. BOURGON. *Application de l'algèbre à la géométrie comprenant la géométrie analytique à deux et à trois dimensions*, troisième édition. Paris.
- Pages 46-54, 56-61, Paris, 1834, pp. 31-41; 8<sup>th</sup> ed. rev. par Darboux, Paris, 1875, pp. 6-38. Analytical discussion only.

# 84 EUCLID'S BOOK ON DIVISIONS OF FIGURES IV

- 1831—H. v. HOLLFFEN, und P. GERWIEN. *Geometrische Analysis*. Berlin, 2 Bde, 1831-1832.

"Theilungen, I, 184, 191; II, 144, 151.

- 1837—G. RITT. *Problèmes d'applications de l'algebre a la géometrie avec les solutions développées*, 2<sup>e</sup> partie. Paris.

Pages 108-109.

- 1840—O. GREGORY. *Hints theoretical, elucidatory and practical, for the use of Teachers of elementary Mathematics and of self-taught students; with especial reference to the first volume of Hutton's course and Simson's Euclid, as Text-Books. Also a selection of miscellaneous tables, and an Appendix on the geometrical division of plane surfaces*. London.

"Appendix: Problems relative to the division of Fields and other surfaces," pp. 158-188; partly taken from Hirsch (1805). See also Euler (1748).

- 1844—DRESER. *Die Theilung der Figuren*. Darmstadt.

This title is taken from E. WOLFFING, *Math. Bucherschatz*, 1903.

- 1847—R. POTTIS. *An appendix to the larger edition of Euclid's Elements of Geometry, containing...Hints for the solution of the Problems...* Cambridge and London.

IV, 91, pp. 72-73.

- 1852—H. CH. DE LA FRÉMOIRE. *Théorèmes et Problèmes de Géométrie élémentaire*, second éd. revue et corrigée par E. Catalan. Paris.

Pages 107-108; 601, par Catalan, Paris, 1879, pp. 160-191.

- 1852—F. RUMMER. *Die Verwandlung und Theilung der Flächen in einer Reihe von Constructions- u. Berechnungs-Aufgaben*. Mit 3 Steintafeln. Heidelberg. 6 + 90 pp.

- 1855—P. KELLAND. "On Superposition." *Transactions of the Royal Society of Edinburgh*, 1885, XXI, 271-273 + 1 pl.

This paper deals, for the most part, with solutions of the following problem proposed to Professor Kelland by Sir John Robison: "From a given square one quarter is cut off, to divide the remaining gnomon into four such parts that they shall be capable of forming a square." In the *Transactions*, 1891, XXXVI, 623, + 2 pls., Robert Brodie has a paper entitled "Professor Kelland's Problem on Superposition."

- 1857—E. CATALAN. *Manuel des Candidats a l'école polytechnique*. Paris, Tome I.

Pages 23, 4: "To divide a circle into two equal parts by means of an arc with its centre  $A$ , on the circumference of the given circles." This is stated by A. REBIÈRE (*Mathématique et Mathématiciens*, 2<sup>e</sup> éd., Paris, 1893, p. 519) under the form: "Quelle doit être la longueur de la longe d'un cheval pour qu'en la fixant au contour d'un pré circulaire l'animal ne puisse tondre que la moitié du pré?"

The solution of this problem leads to a transcendental equation

$$\sin x - x \cos x = \frac{\pi}{2},$$

where  $x$  is the angle under which the points of section of the circumferences are seen from  $A$ . Catalan finds  $x = 109^\circ 11' 18''$ , correct to within a second of arc.

Cf. *L'Intermédiaire des Mathématiciens*, 1914, Question 4, 27, XXI, 5, 6, 90, 115, 180.

- 1863—J. McDOWELL. *Exercises on Euclid and in Modern Geometry*. Cambridge.

No. 157, pp. 145-6; 3rd ed. 1881, p. 118.



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